Building Mathematics and Mutual Understanding. An international conference on occasion of Alfonso Casal's 70th birthday. **E.T.S. de Arquitectura (UPM)**

July 14th-15th 2014

Numerical simulation of a climate model with latent heat of fusion.

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Joint work with L. Tello (UPM) and Ildefonso Díaz (UCM).

POLITÉCNICA

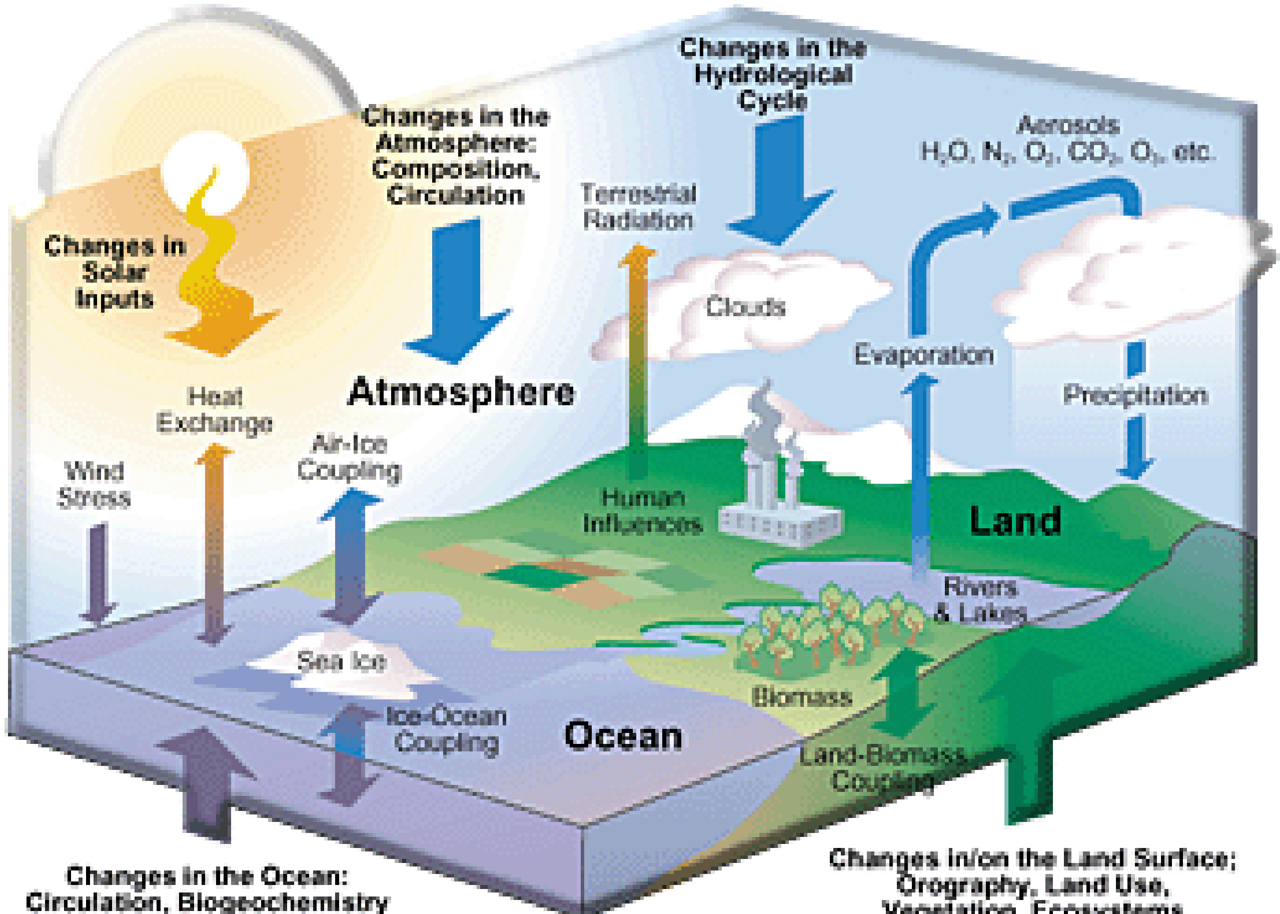
"Ingeniamos el futuro"

Outline

1. Physical problem.
2. Mathematical model.
3. Numerical approximation.
4. Numerical examples.
5. Conclusions and further research.

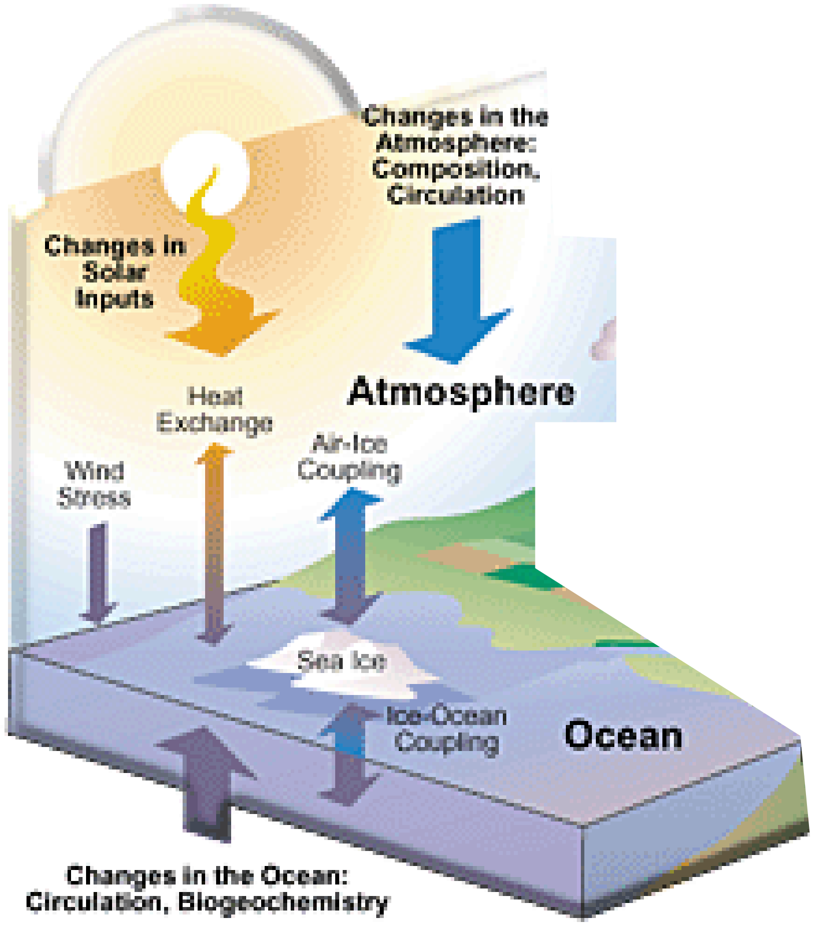
Physical problem

Some processes involved in global climate models:

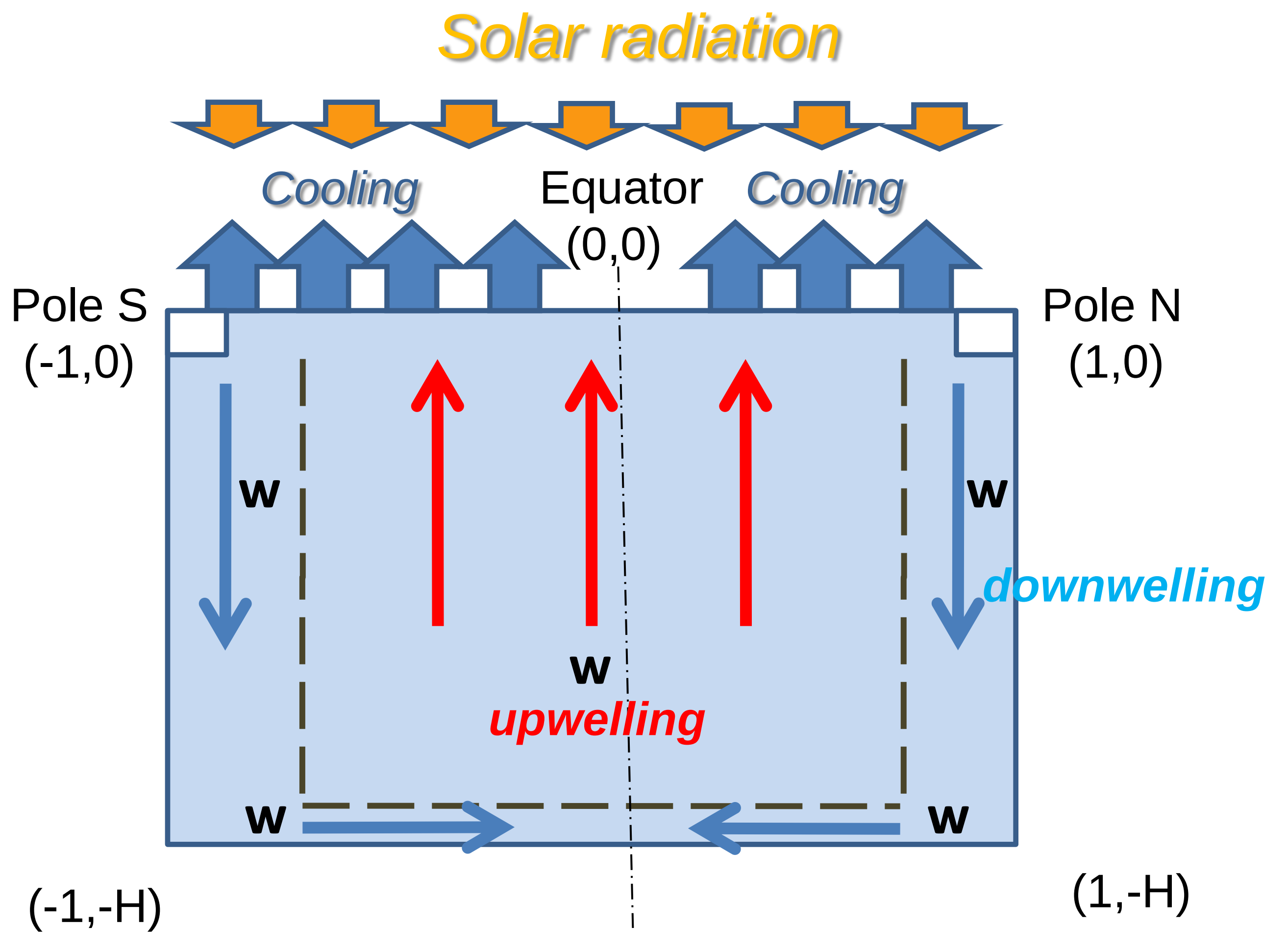


Physical problem

Some processes involved in global climate models:



Physical problem



Mathematical model

THE MODEL (Based on Watts-Morantine [1990])

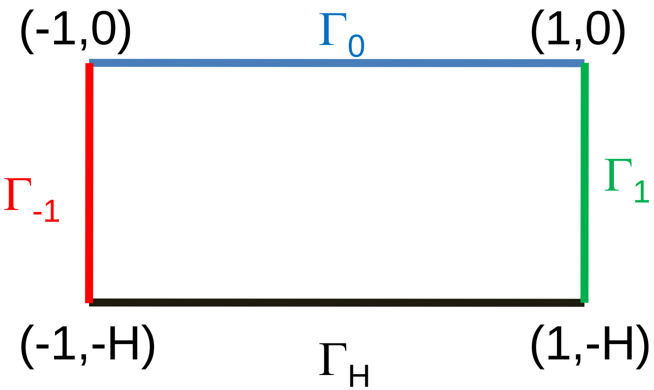
- The model represents the evolution of temperature within an ocean of depth H.
- Spatial variables (x,z): $x = sin$(latitude) and $-z$ (depth).
- Spatial domain $\Omega = (-1,1) \times (0,-H)$
- Boundary: $\Gamma = \Gamma_H \cup \Gamma_0 \cup \Gamma_1 \cup \Gamma_{-1}$

$$\Gamma_H = \left\{ (x,z) \in \bar{\Omega} : \quad z = -H \right\}$$

$$\Gamma_0 = \left\{ (x,z) \in \bar{\Omega} : \quad z = 0 \right\}$$

$$\Gamma_{-1} = \left\{ (x,z) \in \bar{\Omega} : \quad x = -1 \right\}$$

$$\Gamma_1 = \left\{ (x,z) \in \bar{\Omega} : \quad x = 1 \right\}$$



•The model considers the average temperature over each parallel as the unknown.

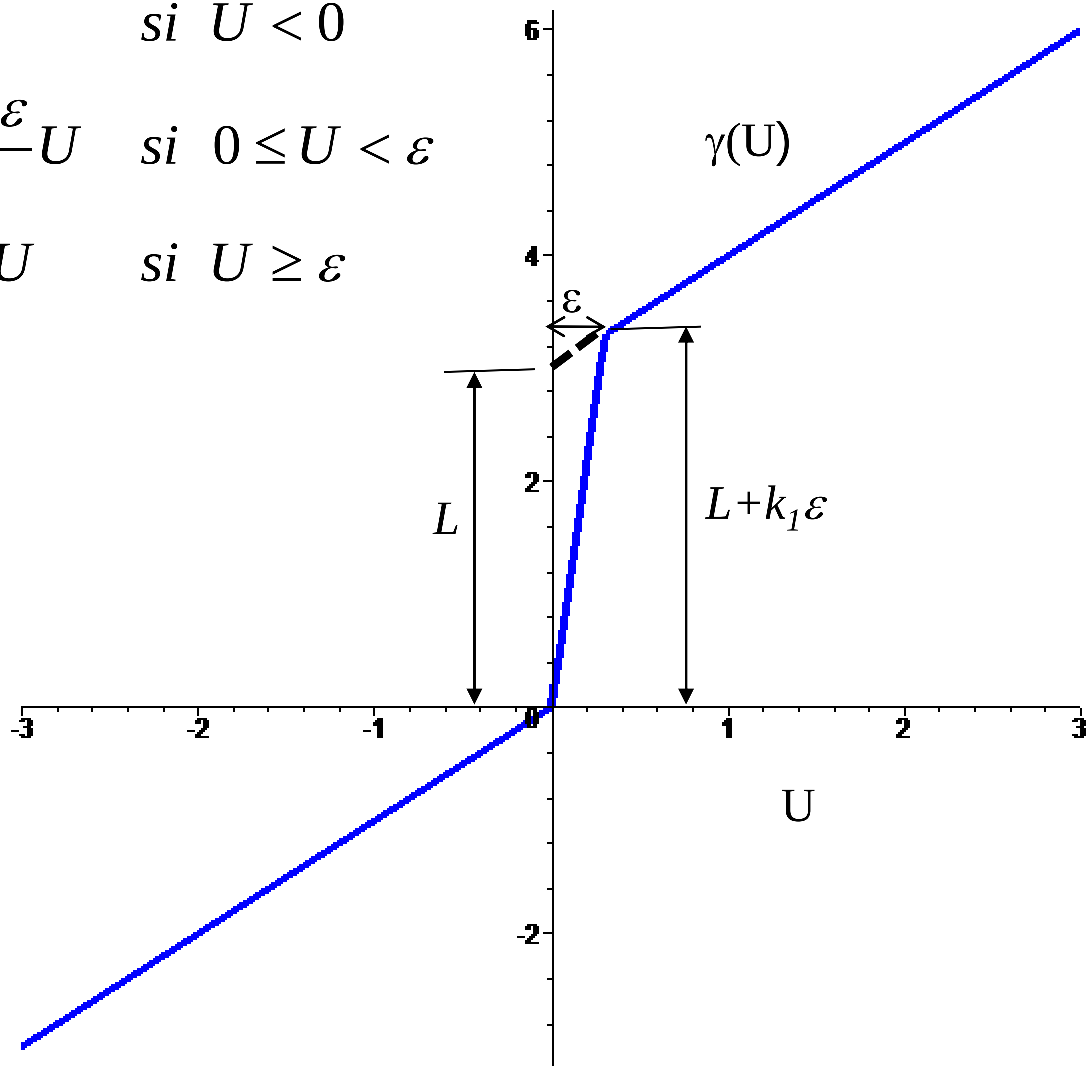
Mathematical model

Deep Ocean Model (DOM) :

$$\gamma(U)_t - \left(\frac{K_H}{R^2}(1-x^2)U_x\right)_x - K_V U_{zz} + \omega U_z = 0 \quad \text{in } \Omega\times(0,T)$$

U: temperature,
ω: vertical velocity,
K_V: vertical diffusivity,
K_H: horizontal diffusivity,
R: radius of the Earth.

$$\gamma(U) = \begin{cases} k_2 U & si \quad U < 0 \\ \dfrac{L + k_1 \varepsilon}{\varepsilon} U & si \quad 0 \le U < \varepsilon \\ L + k_1 U & si \quad U \ge \varepsilon \end{cases}$$



Mathematical model

Boundary condition for ocean bottom

$$\omega x U_x + K_v U_z = 0, \quad on \quad \Gamma_H \times (0,T)$$

Boundary condition for upper boundary: Energy Balance Model (EBM)

$$Du_t - \frac{DK_{H_0}}{R^2}\left((1-x^2)^{\frac{p}{2}} |u_x|^{p-2} u_x\right)_x + Bu + C + K_V \frac{\partial U}{\partial n} + \omega x u_x \in \frac{1}{\rho c} QS(x)\beta(u)$$

$$on \quad \Gamma_0 \times [0,T]$$

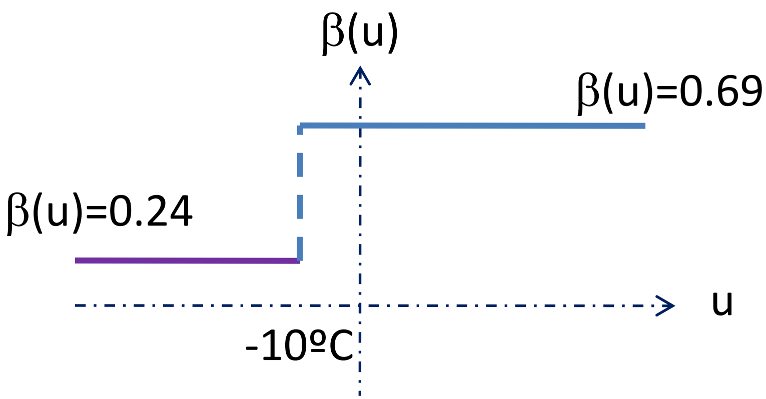
u: temperature,
ω: velocity,
K_v: vertical diffusivity,
K_{H0}: horizontal diffusivity,
R: radius of the Earth.

D: thickness mixed layer,
ρ: density,
c: specific heat coeff.,
β(u): coalbedo,
Q: solar constant,
Bu+C: cooling term,
S(x): insolation.

Mathematical model

1) We consider the case p=3 (Stone, 1972)

2) We use the coalbedo, $\beta(u)$, (Budyko model)



Mathematical model

$$\begin{cases} \gamma(U)_t - \left(\dfrac{K_H}{R^2}(1-x^2)U_x\right)_x - K_V U_{zz} + \omega U_z = 0 \quad \text{in } \Omega\times(0,T), \\ \omega x U_x + K_V U_z = 0 \quad \text{in } \Gamma_H \times (0,T), \\ Du_t - \dfrac{DK_{H_0}}{R^2}\left((1-x^2)^{3/2}\left|u_x\right|u_x\right)_x + K_V\dfrac{\partial U}{\partial n} + \omega x u_x + Bu + C \in \dfrac{1}{\rho c} QS(x)\beta(u) \\ \text{on } \Gamma_0 \times (0,T), \\ U_x = 0, \quad on \quad \Gamma_{-1}\times[0,T] \quad , \\ U_x = 0, \quad in \quad \Gamma_1\times[0,T] \quad , \\ U(x,z,0) = U_0(x,z), \quad in \quad \Omega \quad , \\ u(x,0) = u_0(x), \qquad in \quad \Gamma_0 \quad . \end{cases}$$

Final system

A. Hidalgo. Conference Honour
Alafonso Casal. July 14-15. 2014.

Mathematical model

The term:

$$\frac{K_v}{D}\frac{\partial U}{\partial n}$$

Stands for the coupling atmosphere-ocean in the sense of analyzing the influence of the ocean temperature in the atmosphere.

In this work we shall show results with and without this term.

Mathematical model

Some references about global climate EBM models with or without deep ocean effect :

- Watts&Morantine (1990),
- Xu (1990),
- Hetzer (1990),
- Kim, North & Huang (1992),
- Díaz (1993),
- Schmidt (1994),
- Díaz-Hernández-Tello (1997),
- Arcoya-Díaz-Tello (1998),
- Hetzer (2000),
- Díaz-Tello (2007),
- Bermejo et al (2008),
- Hidalgo-Tello (2010),
- …

Numerical approximation

We rewrite this problem as advection-reaction-diffusion equations, both for the upper boundary EBM and for the DOM.

EBM:

$$u_t - \left(f\left(x, u(x,t), u_x(x,t)\right)\right)_x = \sigma(x, u(x,t), \frac{\partial U}{\partial n}(x,0,t))$$

with the flux:

$$f\left(x, u(x,t), u_x(x,t)\right) := \frac{K_{H_0}}{R^2}(1-x^2)^{3/2}\left|u_x(x,t)\right|u_x(x,t) - \frac{w}{D}xu(x,t)$$

and the source term:

$$\sigma(x, u, \frac{\partial U}{\partial n}) := \frac{1}{D}(-C + \frac{Q}{\rho c}S(x)\beta(u) + (\omega + x\omega_x - B)u(x,t) - K_V\frac{\partial U}{\partial n})$$

Numerical approximation

DOM:

$$\gamma(U(x,z,t))_t - (F(x,U_x(x,z,t)))_x - (G(U(x,z,t),U_z(x,z,t)))_z = \Xi(x,U(x,z,t)),$$

with the fluxes :

$$F\left(x,U_x(x,z,t)\right) := \frac{K_H}{R^2}\left(1-x^2\right)U_x(x,z,t),$$
$$G(U(x,z,t),U_z(x,z,t)) := K_V U_z(x,z,t) - wU(x,z,t),$$

and the source term:

$$\Xi(x,U(x,z,t)) := \omega_z U(x,z,t).$$

<u>Numerical approach</u>: finite volume method with Weighted Essentially Non-Oscillatory (WENO) reconstruction in space and third-order Runge-Kutta TVD for time integration.

For each time step, we compute a numerical solution of the EBM model equation for each cell u_i^{n+1}

$$Du_t - \frac{DK_{H_0}}{R^2}\left((1-x^2)^{\frac{p}{2}}\left|u_x\right|^{p-2}u_x\right)_x + Bu + C + K_V\frac{\partial U}{\partial n} + \omega x u_x \in QS(x)\beta(u)$$

$$on \quad \Gamma_0 \times [0,T]$$

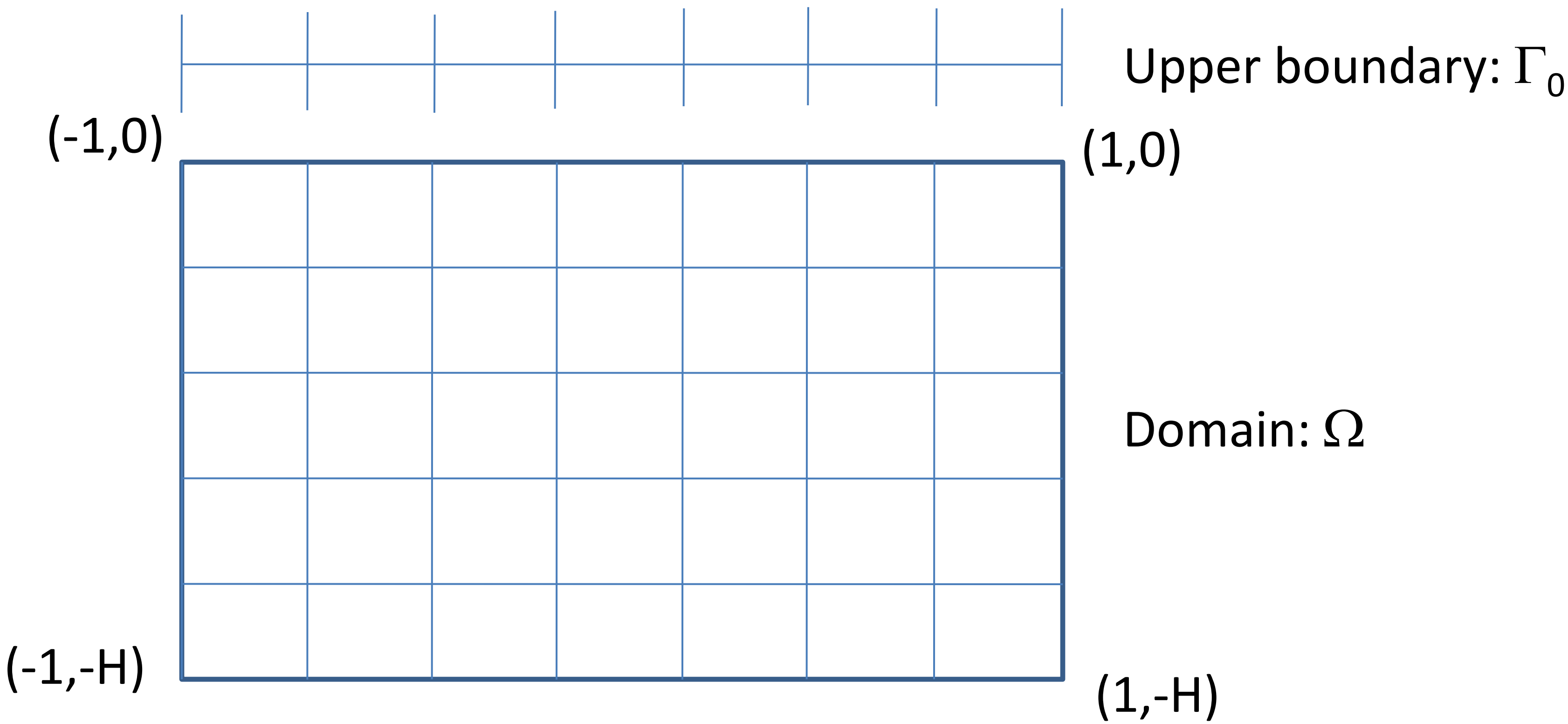
then we use u_i^{n+1} as a Dirichlet boundary condition for the DOM to obtain $\gamma_{i,j}^{n+1}$

$$\gamma(U)_t - \left(\frac{K_H}{R^2}(1-x^2)U_x\right)_x - K_V U_{zz} + \omega U_z = 0 \quad \text{in}\ \Omega\times(0,T)$$

finally we apply an iterative solver for nonlinear equations (Newton+bisection) to obtain

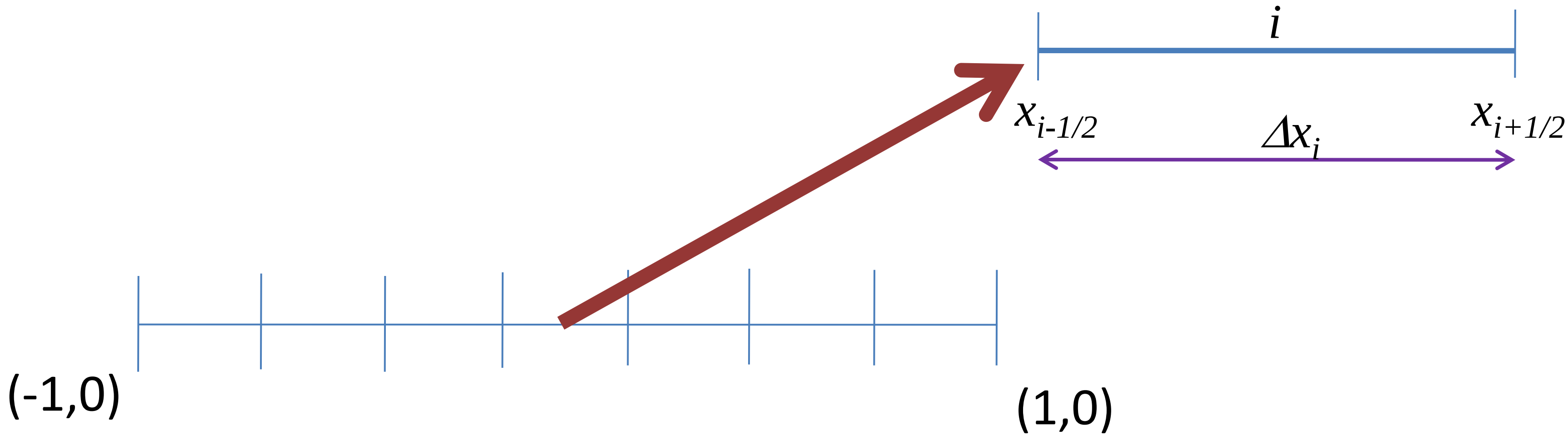
$$U_{i,j}^{n+1}$$

The finite volume framework



The finite volume framework

We discretize the 1D domain [-1,1] in N_x control volumes of length $\Delta x_i = x_{i+1/2} - x_{i-1/2}$.



The finite volume framework

We integrate the equation dividing by the length of the control volume to obtain the following ordinary differential equation (ODE)

$$\frac{du_i(t)}{dt} = \frac{1}{\Delta x_i}\left(f_{i+1/2} - f_{i-1/2} \right) + \sigma_i(t) \equiv l_i(u(t)),$$

where

$$u_i(t) = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} u\left(x,t\right)dx$$ integral average of the unknown,

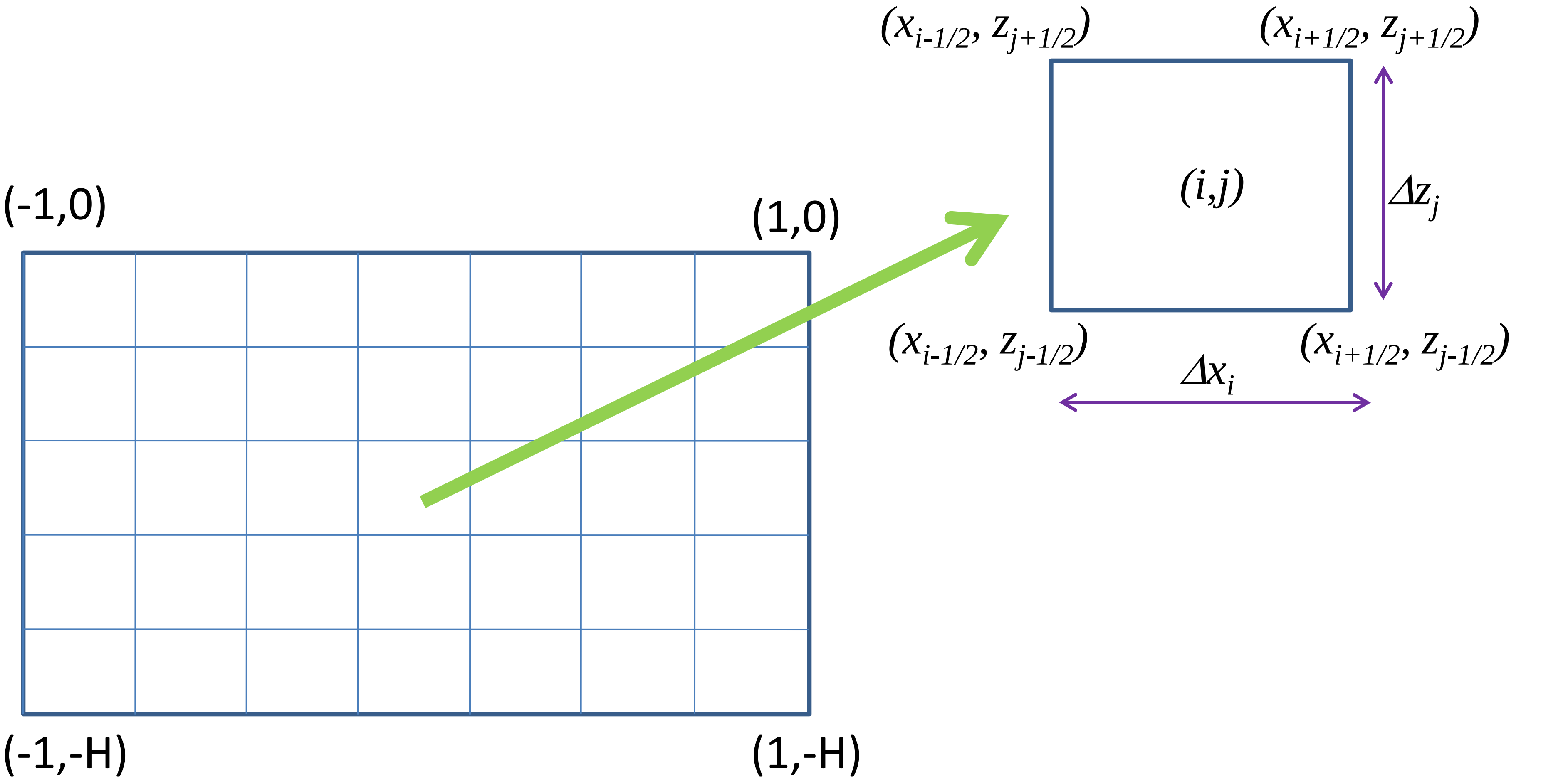
$$f_{i+1/2} = f\left(\mathrm{x}, u\left(x_{i+1/2},t\right), u_x\left(x_{i+1/2},t\right)\right)$$ right intercell numerical flux,

$$\sigma_i(t) = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} \sigma(x,u,\frac{\partial U}{\partial n})dx$$ integral average of the source term.

The finite volume framework

We discretize the 2D domain $[-1,1] \times [0,-H]$ in $N_x N_z$ control volumes of area $\Delta x_i \times \Delta z_j$

$\Delta x_i = x_{i+1/2} - x_{i-1/2}$, $\Delta z_j = z_{j+1/2} - z_{j-1/2}$



The finite volume framework

We integrate the equation dividing by the area of the control volume to obtain the following ordinary differential equation (ODE)

$$\frac{d\gamma_{i,j}}{dt} = \frac{1}{\Delta x_i}\left(F_{i+1/2,j} - F_{i-1/2,j}\right) + \frac{1}{\Delta z_j}\left(G_{i,j+1/2} - G_{i,j-1/2}\right) + \Gamma_{i,j} \equiv L_{i,j}$$

where

$$\gamma_{i,j} = \frac{1}{\Delta x_i \Delta z_j}\int_{z_{j-1/2}}^{z_{j+1/2}}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} \gamma\left(U(x,z,t)\right)dx\right)dz$$

integral average of the unknown,

$$F_{i+1/2,j} = \frac{1}{\Delta z_j}\int_{z_{j-1/2}}^{z_{j+1/2}} F\left(x_{i+1/2}, U_x\left(x_{i+1/2}, z, t\right)\right)dz,$$

Spatial integral average of intercell fluxes,

$$G_{i,j+1/2} = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} G\left(U\left(x, z_{j+1/2}, t\right), U_z\left(x, z_{j+1/2}, t\right)\right)dx,$$

$$\Gamma_{i,j}(t) = \frac{1}{\Delta x_i \Delta z_j}\int_{z_{j-1/2}}^{z_{j+1/2}}\left(\int_{x_{i-1/2}}^{x_{i+1/2}} \Xi(x, U(x,z,t))dx\right)dz,$$

integral average of the source term.

Runge Kutta TVD

EBM:

$$u^{k,1} = u^{n} + \Delta t l(u^{n}), \qquad u^{k,2} = \frac{3}{4}u^{n} + \frac{1}{4}u^{k,1} + \frac{1}{4}\Delta t l(u^{k,1}),$$

$$u^{n+1} = \frac{1}{3}u^{n} + \frac{2}{3}u^{k,2} + \frac{2}{3}\Delta t l(u^{k,2}).$$

DOM:

$$\gamma^{k,1} = \gamma^{n} + \Delta t L(\gamma^{n}),$$

$$\gamma^{k,2} = \frac{3}{4}\gamma^{n} + \frac{1}{4}\gamma^{k,1} + \frac{1}{4}\Delta t L\left(\gamma^{k,1}\right),$$

$$\gamma^{n+1} = \frac{1}{3}\gamma^{n} + \frac{2}{3}\gamma^{k,2} + \frac{2}{3}\Delta t L\left(\gamma^{k,2}\right).$$

WENO reconstruction

EBM

1) For intercell fluxes

For an order of accuracy r we have r candidate stencils each one of them with r cells

$$\{S_{i-r+1}, S_{i-r+2},\ldots,S_i\}, \{S_{i-r+2}, S_{i-r+3},\ldots,S_{i+1}\}, \ldots, \{S_i, S_{i+1},\ldots,S_{i+r-1}\}$$

For each stencil we consider a (r-1)th degree interpolating polynomial

$$p_l(x), \quad l = 0,\cdots,r-1$$

Each one of the polynomials considered must be conservative:

$$\frac{1}{\Delta x_k}\int_{S_k} p_l(x)dx = u_k(t), \qquad 0 \le l \le r-1, \quad 0 \le k \le r-1$$

WENO reconstruction

The WENO procedure defines the reconstructed values

$$u(x_{i+1/2},t), \quad u_x(x_{i+1/2},t)$$

as a convex combination of the rth-order accurate values of all polynomials taken with positive nonlinear weights, ω_k.

$$u(x_{i+1/2},t) = \sum_{k=0}^{r-1} \omega_k u_{i+1/2}^{(k,0)}, \quad u_x(x_{i+1/2},t) = \sum_{k=0}^{r-1} \omega_k u_{i+1/2}^{(k,1)}$$

where

$$\alpha_k = \frac{d_k}{(\varepsilon + \beta_k)^p}; \quad \omega_k = \frac{\alpha_k}{\sum_{j=0}^{r-1} \alpha_j} \quad (k = 0,1,\ldots,r-1) \qquad (\varepsilon = 10^{-6}; \quad p = 2)$$

with the smoothness indicators given by

$$\beta_k = \sum_{m=0}^{r-1} \int_{x_{i-1/2}}^{x_{i+1/2}} \left(\frac{d^m}{dx^m} p_k(x) \right)^2 \Delta x^{2m-1} dx \quad (k = 0,\ldots,r-1)$$

And the optimal weights (see Dumbser, Enaux, Toro, 2008)

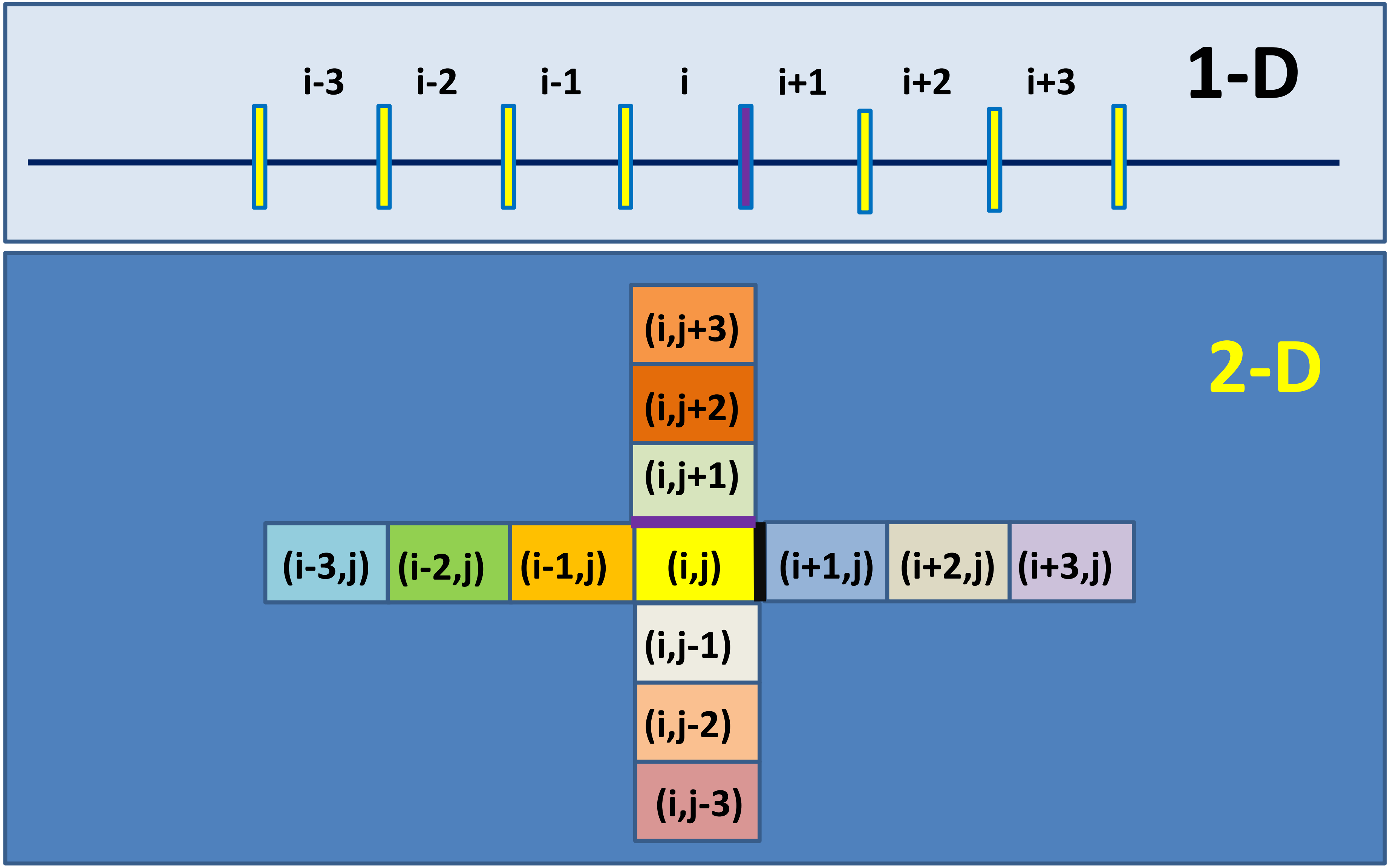
$$d_0 = d_3 = 10^0; \quad d_1 = d_2 = 10^{10}$$

WENO reconstruction

In this work we have used r=4. Therefore, the candidate stencils are:

$$\{S_{i-3}, S_{i-2}, S_{i-1}, S_i\}, \{S_{i-2}, S_{i-1}, S_i, S_{i+1}\},$$
$$\{S_{i-1}, S_i, S_{i+1}, S_{i+2}\}, \{S_i, S_{i+1}, S_{i+2}, S_{i+3}\}.$$

WENO reconstruction



WENO reconstruction

$$u(x_{i+1/2},t) = \sum_{k=0}^{3} \omega_k u_{i+1/2}^{(k,0)}$$

$$u_{i+1/2}^{(k,0)} = \sum_{j=-3}^{3} C_j u_{i+j}(t)$$

k	C_{-3}	C_{-2}	C_{-1}	C_0	C_1	C_2	C_3
0	0	0	0	1/4	13/12	-5/12	1/12
1	0	0	-1/12	7/12	7/12	-1/12	0
2	0	1/12	-5/12	13/12	1/4	0	0
3	-1/4	13/12	-23/12	25/12	0	0	0

WENO reconstruction

$$u_x(x_{i+1/2},t)=\sum_{k=0}^{r-1}\omega_k u_{i+1/2}^{(k,1)}$$

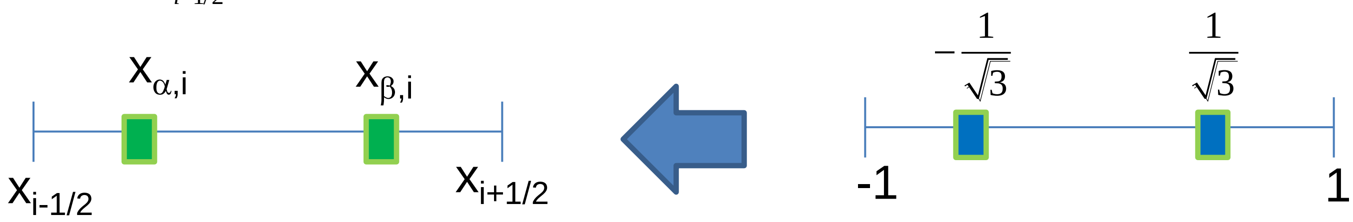
$$u_{i+1/2}^{(k,1)}=\frac{1}{\Delta x}\sum_{j=-3}^{3}D_j u_{i+j}(t)$$

k	D_{-3}	D_{-2}	D_{-1}	D_0	D_1	D_2	D_3
0	0	0	0	-11/12	9/12	3/12	-1/12
1	0	0	1/12	-15/12	15/12	-1/12	0
2	0	1/12	-3/12	11/12	9/12	0	0
3	-11/12	45/12	-69/12	35/12	0	0	0

WENO reconstruction

2) For source term

$$\sigma_i(t) = \frac{1}{\Delta x_i} \int_{x_{i-1/2}}^{x_{i+1/2}} (\sigma(x,t))dx \approx \frac{1}{2}\Big(\sigma\big(u\big(x_{\alpha,i},t\big)\big) + \sigma\big(u\big(x_{\beta,i},t\big)\big)\Big)$$



where the Gaussian points $x_{\alpha,i}$, $x_{\beta,i}$ are

$$x_{\alpha,i} = \frac{1}{2}\left(x_{i+1/2} + x_{i-1/2}\right) - \frac{\sqrt{3}}{6}\left(x_{i+1/2} - x_{i-1/2}\right)$$

$$x_{\beta,i} = \frac{1}{2}\left(x_{i+1/2} + x_{i-1/2}\right) + \frac{\sqrt{3}}{6}\left(x_{i+1/2} - x_{i-1/2}\right)$$

and the values $u(x_{\alpha,i},t)$ and $u(x_{\beta,i},t)$ are obtained via WENO procedure.

WENO reconstruction

$$u\left(x_{\alpha,i},t\right)=\sum_{j=-3}^{3}E_j u_{i+j}(t)$$

$$r=\sqrt{3}$$

k	E_{-3}	E_{-2}	E_{-1}	E_0	E_1	E_2	E_3
0	0	0	0	1+65r/216	-35r/72	17r/72	-11r/216
1	0	0	11r/216	1+7r/72	-13r/72	7r/216	0
2	0	-7r/216	13r/72	1-7r/72	-11r/216	0	0
3	11r/216	-17r/12	35r/72	1-65r/216	0	0	0

$$u\left(x_{\beta,i},t\right)=\sum_{j=-3}^{3}F_j u_{i+j}(t)$$

$$r=\sqrt{3}$$

k	F_{-3}	F_{-2}	F_{-1}	F_0	F_1	F_2	F_3
0	0	0	0	1-65r/216	35r/72	-17r/72	11r/216
1	0	0	-11r/216	1-7r/72	13r/72	-7r/216	0
2	0	7r/216	-13r/72	1+7r/72	11r/216	0	0
3	-11r/216	17r/12	-35r/72	1+65r/216	0	0	0

WENO reconstruction

DOM

In the 2D DOM we need to compute the intercell numerical fluxes $F_{i+1/2,j}$ and $G_{i,j+1/2}$ from cell averages.

$$U_{i,j}(t) = \frac{1}{\Delta x_i \Delta z_j} \int_{z_{j-1/2}}^{z_{j+1/2}} \left(\int_{x_{i-1/2}}^{x_{i+1/2}} U(x,t)\,dx \right) dz.$$

We carry it out using two one-dimensional WENO reconstructions.

Numerical approximation

Some references about the numerical approximation :

- **Toro-Hidalgo (2009)**
- **Shu-Gottlieb (1998),**
- **Toro-Titarev (2004, 2005),**
- **Dumbser-Enaux-Toro (2009).**

UNITS SCALING:

Values of physical parameters (Watts & Morantine, 1990)

K_{H0}=1.2E+12 m^2/yr.

D=60

K_H=2.0E+10 m^2/yr.

K_V=2000 m^2/yr.

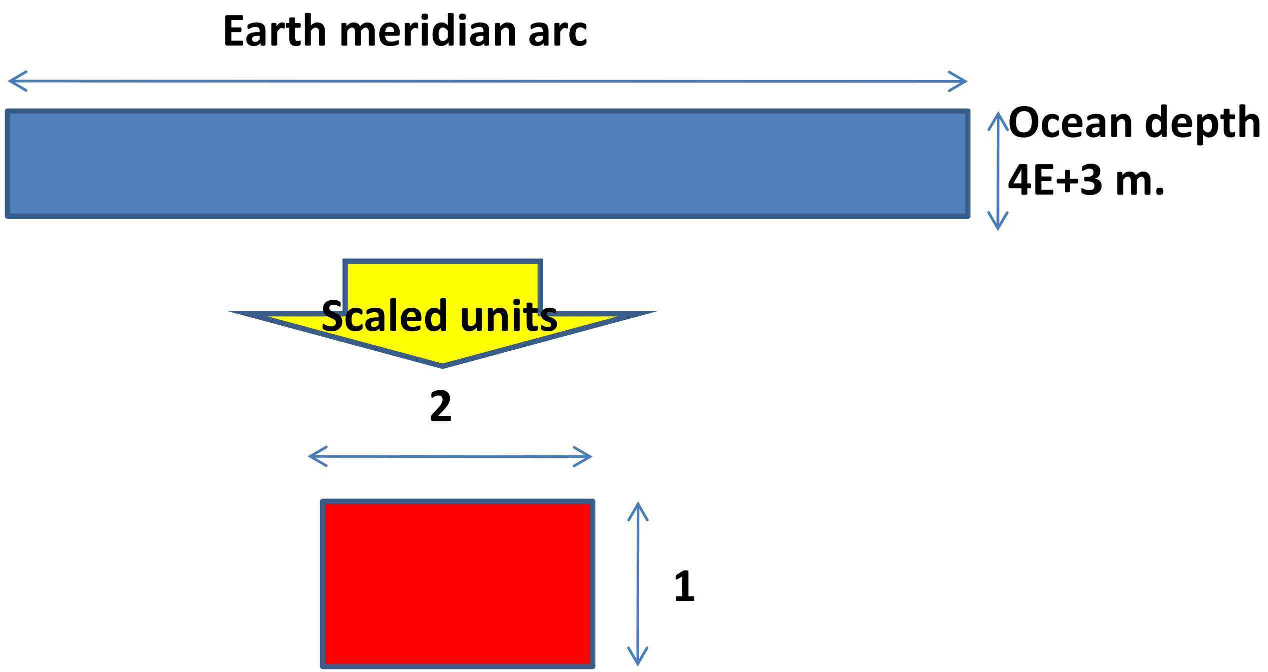
ω=4 m/yr.

Q=340

c=1

ρ=1

S=1-1/2 $P_2(x)$



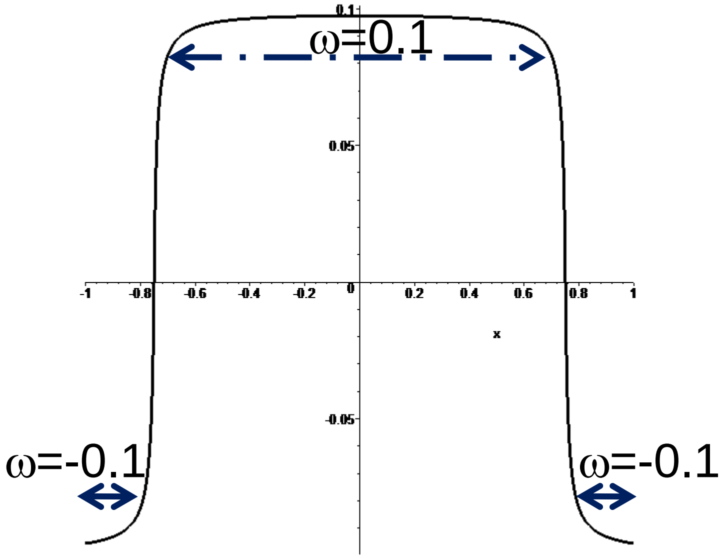
Numerical example without latent heat: $\gamma(U)=U$

Physical parameters:

Parameter	Scaled Value
K_H	0.049
K_{H0}	0.555×10^{-3}
K_V	0.0125
C, B	190, 2
c, ρ	1, 1
Q	340
D	60

$$S = 1-\frac{1}{2}P_2(x)$$

$$\omega(x,z) = W(x) = \frac{10(x+0.75)(x-0.75)}{(0.1+10|x+0.75|)(0.1+10|x-0.75|)}$$



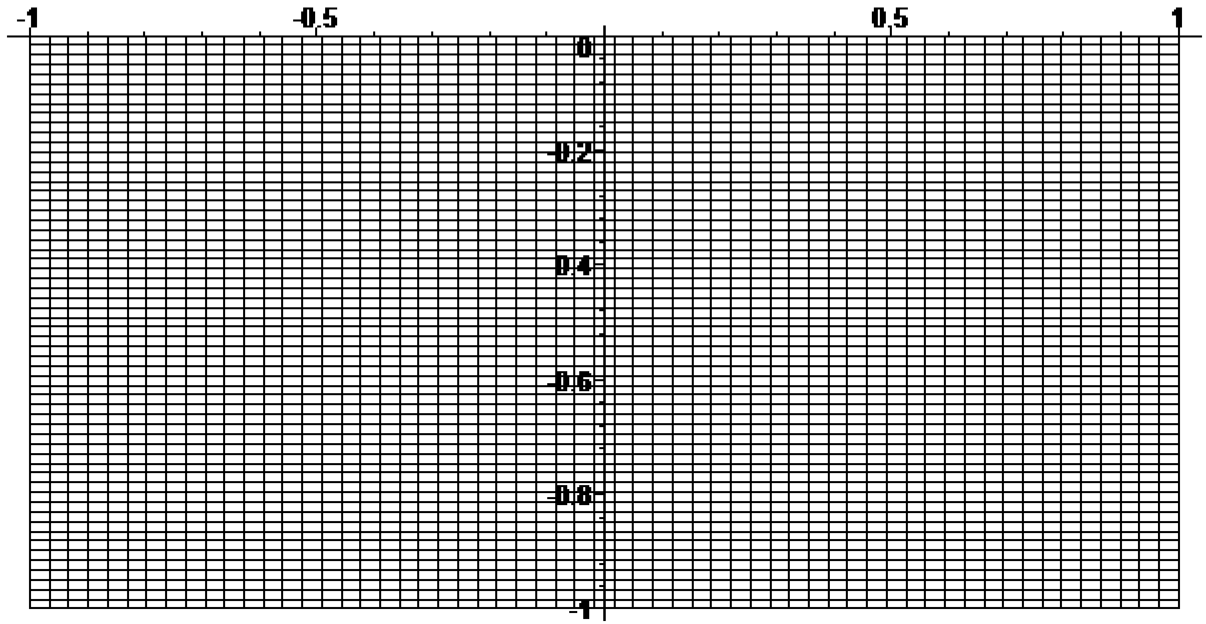
Space and time discretization:

$$\Delta x = 2/60; \qquad \Delta z = 1/60$$

$$\Delta t = \min\left(\alpha\Delta x^2\left(\left(1-x^2\right)KH\right)^{-1}, \alpha\Delta z^2\left(KV\right)^{-1}, \alpha\Delta x^2\left(\left(1-x^2\right)KH0\left|\frac{du}{dx}\right|\right)^{-1}\right), (\alpha = 0.3)$$

Numerical example

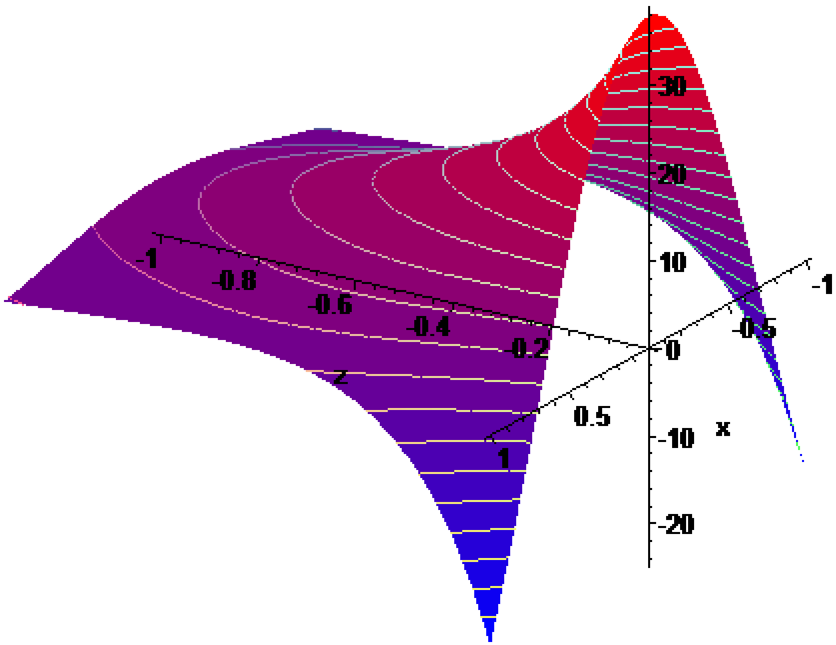
Spatial mesh used (60 x 60 cells)



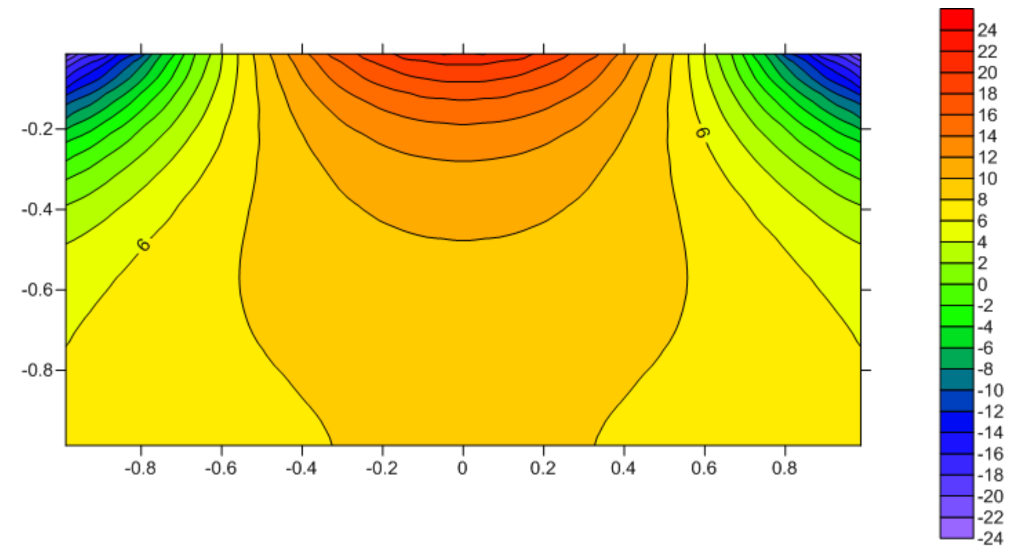
Numerical example

Initial condition:

$$U(x,z,0)=18e^{-x^2-z^2}+80e^{-x^2}-60$$



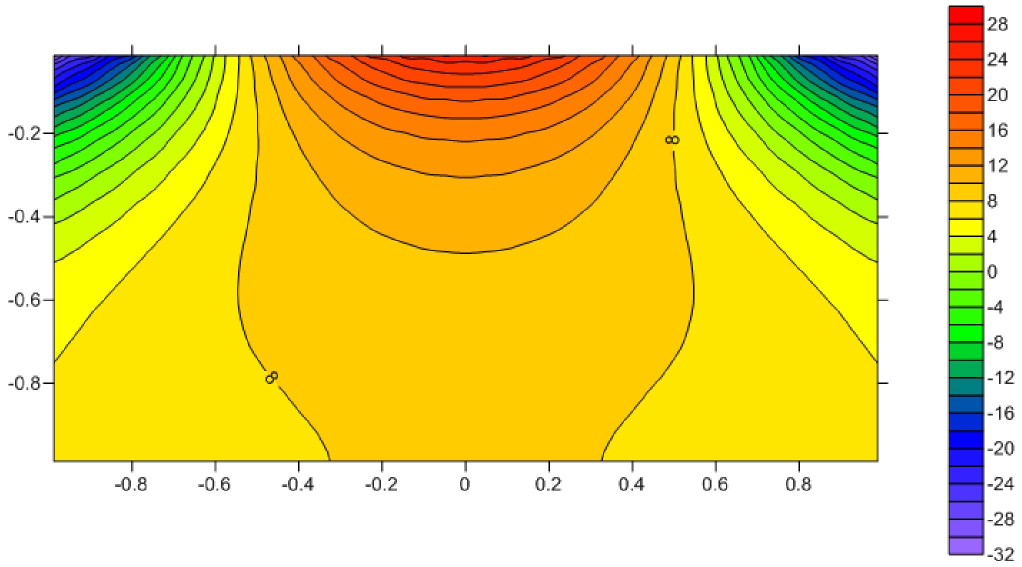
Numerical example



DOM solution.
Output time = 1.0

<u>WITH</u> influence of deep ocean on atmosphere.

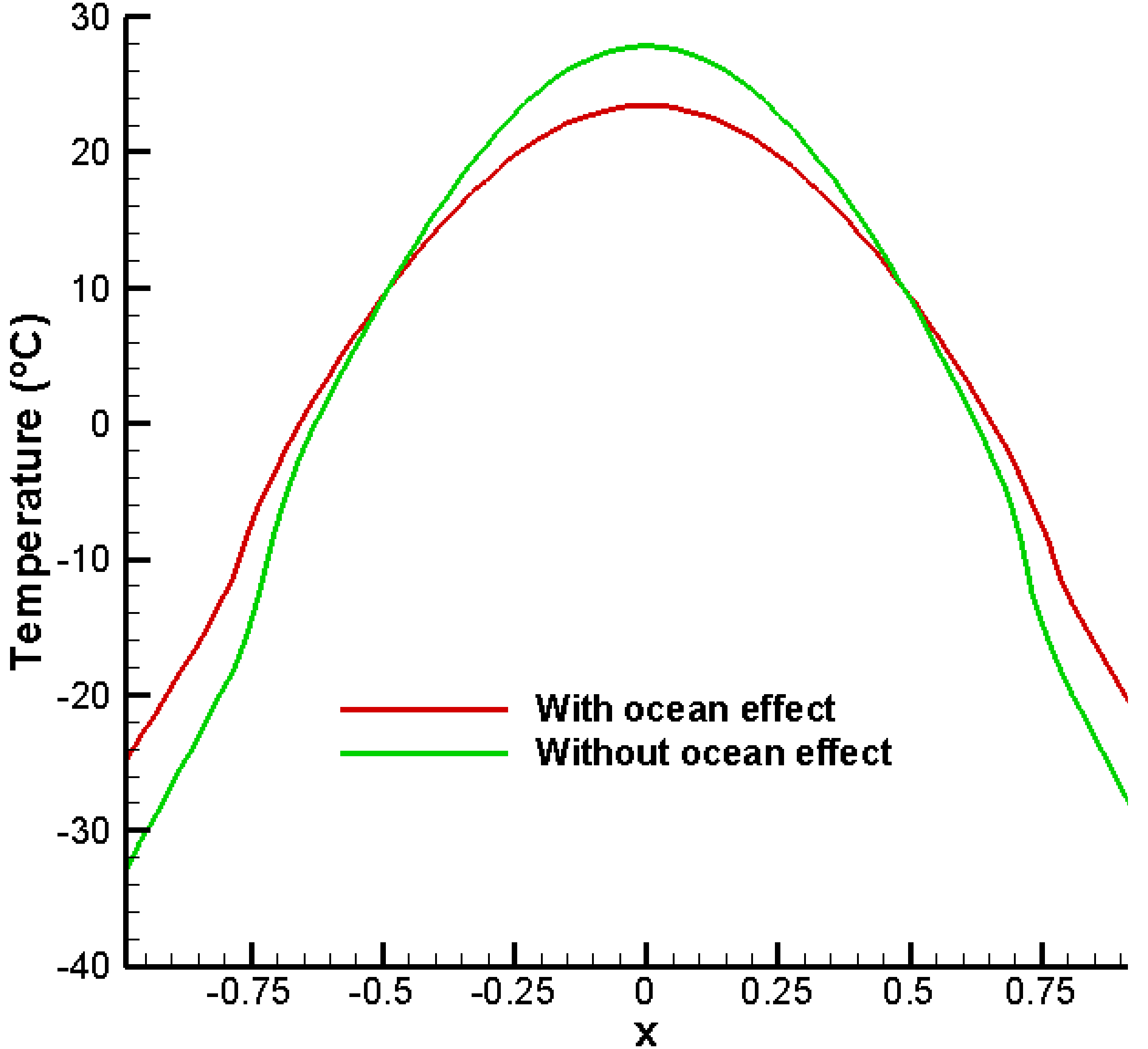
$$\frac{K_v}{D}\frac{\partial U}{\partial n} \neq 0$$



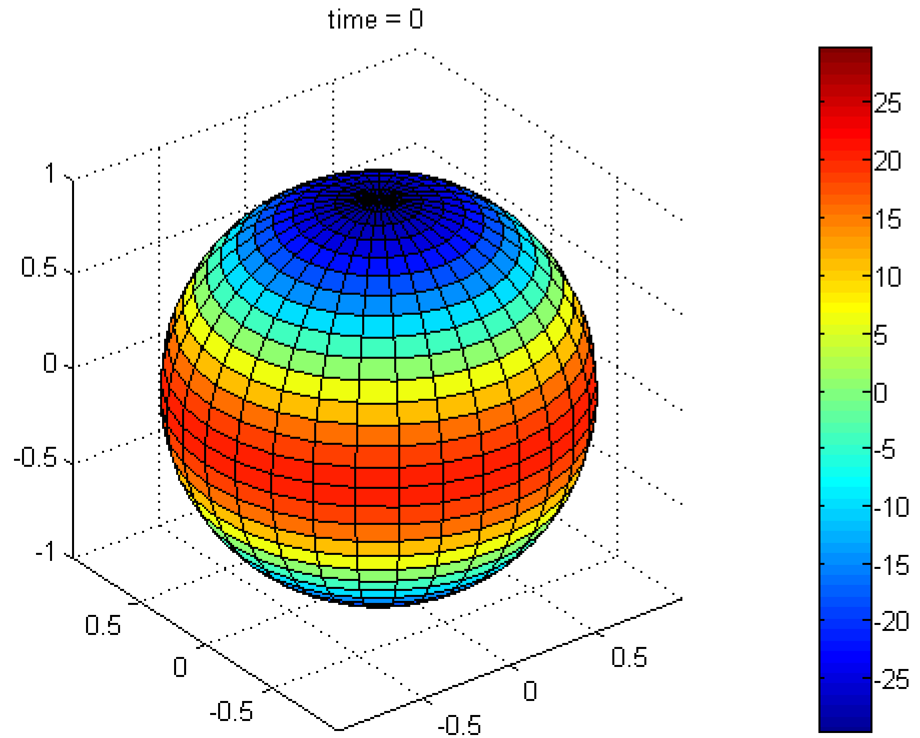
<u>WITHOUT</u> influence of deep ocean on atmosphere.

$$\frac{K_v}{D}\frac{\partial U}{\partial n} = 0$$

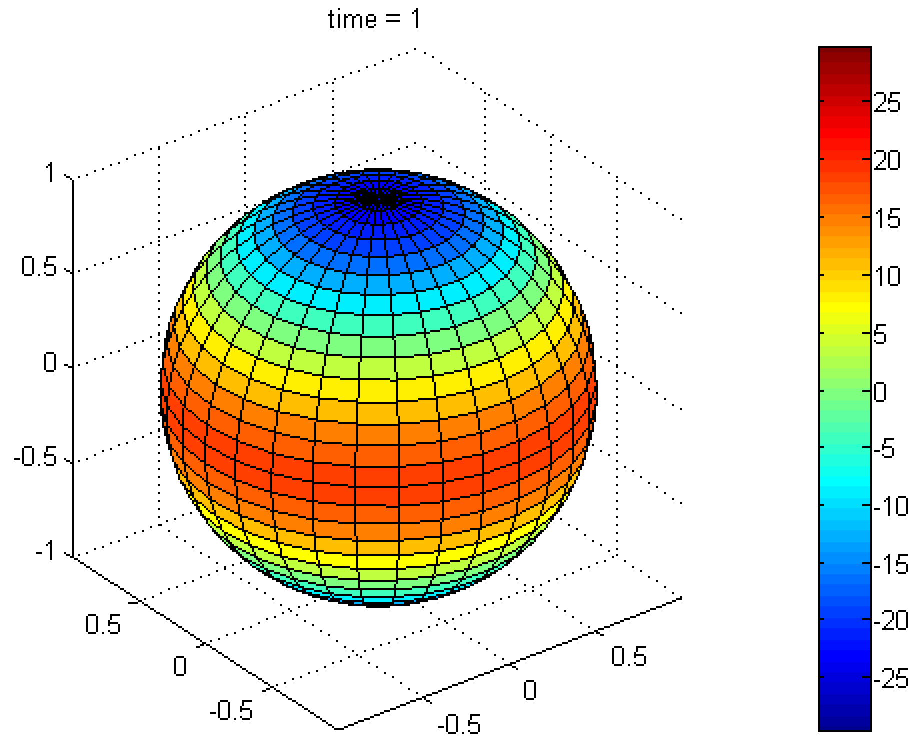
Numerical example



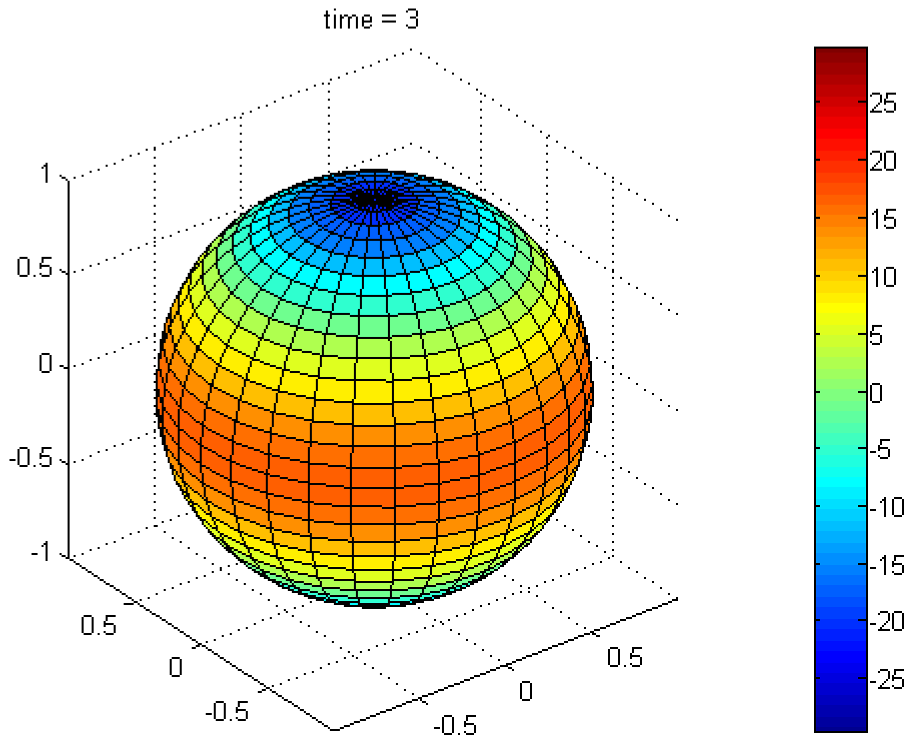
Temperature distribution on the surface of the ocean: assumed constant on each parallel.



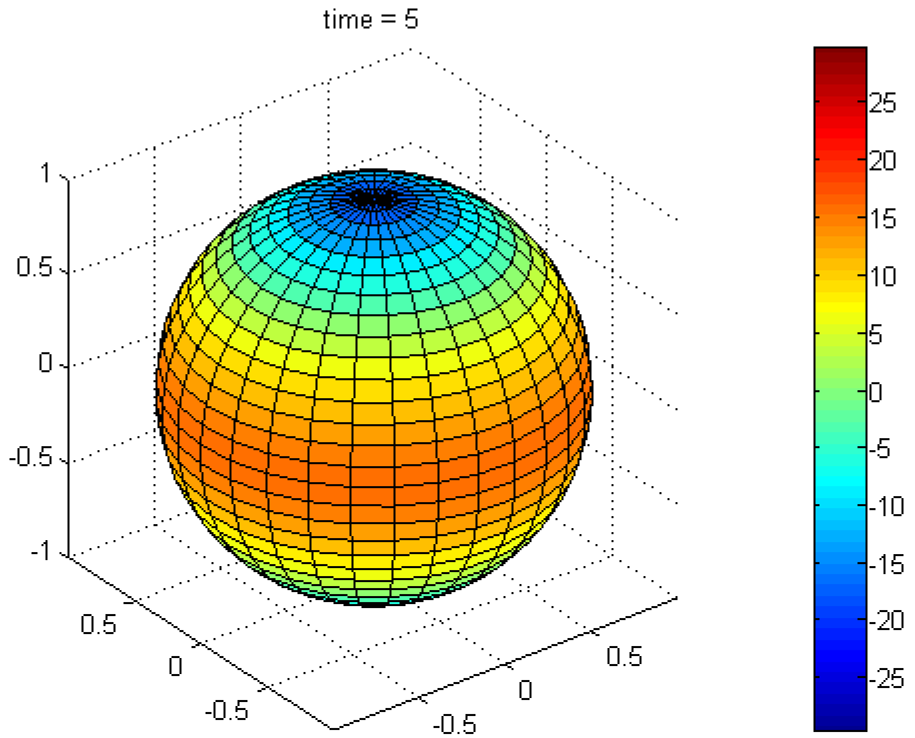
Temperature distribution on the surface of the ocean: assumed constant on each parallel.



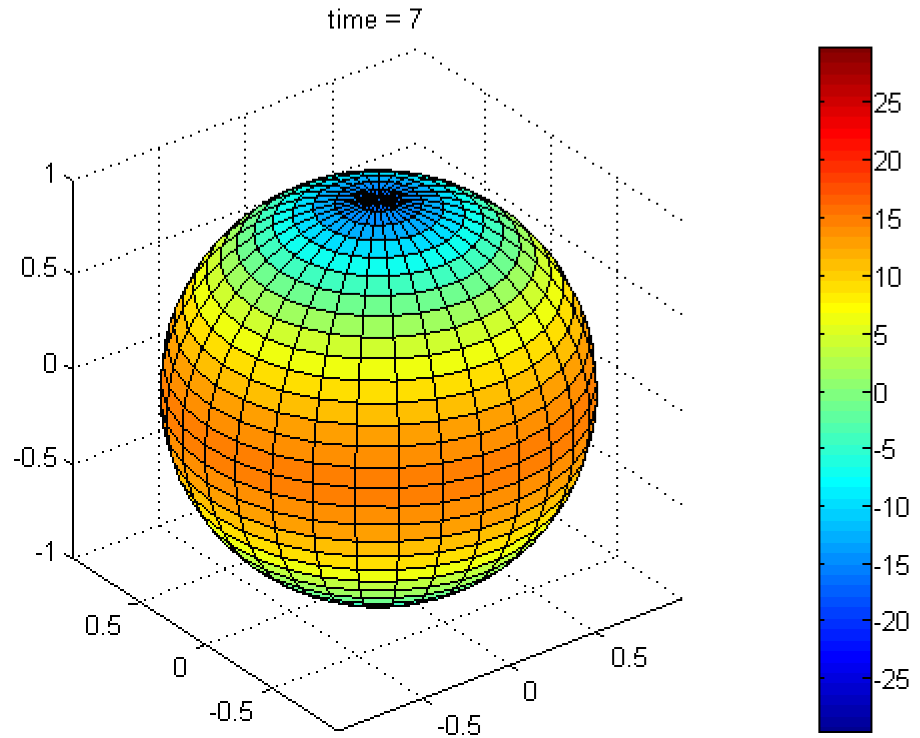
Temperature distribution on the surface of the ocean: assumed constant on each parallel.



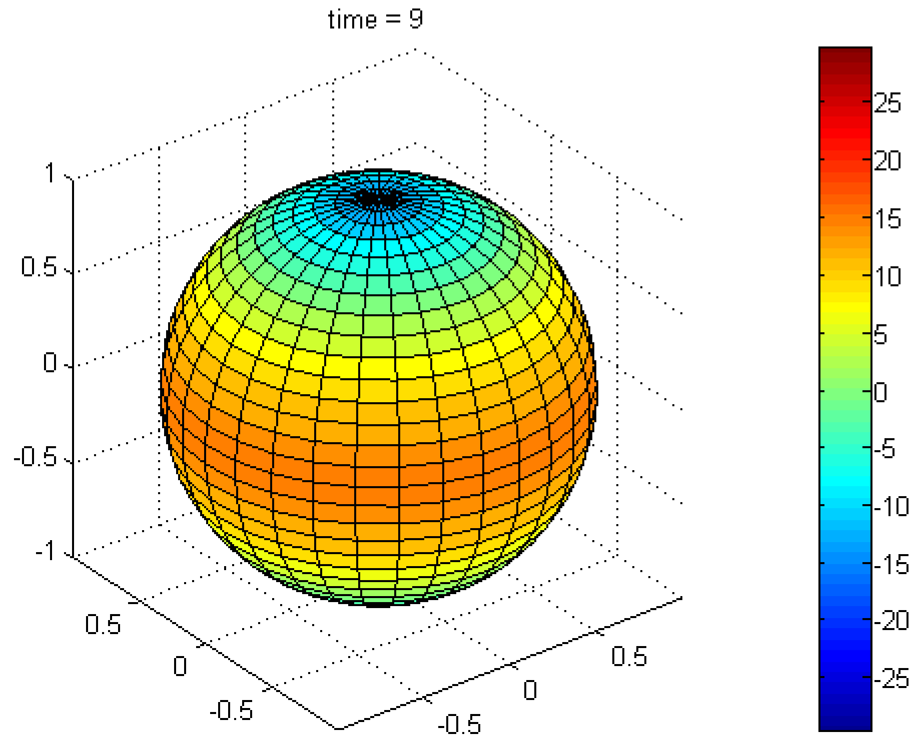
Temperature distribution on the surface of the ocean: assumed constant on each parallel.



Temperature distribution on the surface of the ocean: assumed constant on each parallel.



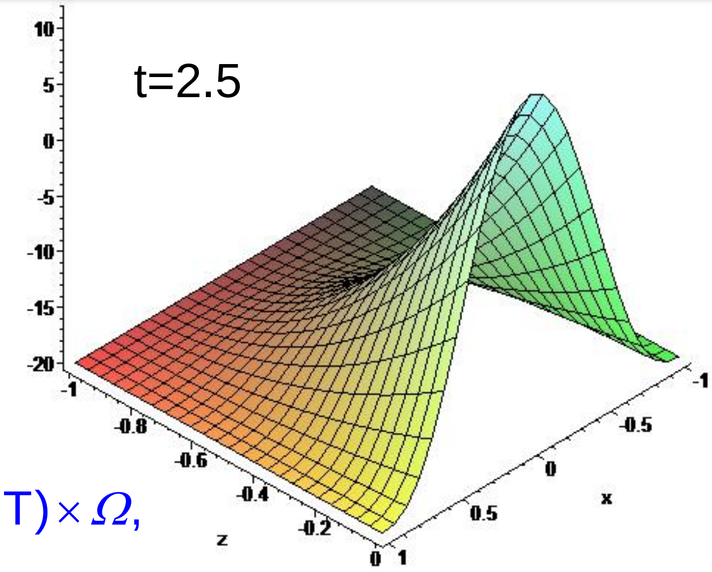
Temperature distribution on the surface of the ocean: assumed constant on each parallel.



Validation of the numerical scheme

We consider the function:

$$U(x,z,t)=\frac{(100(1-x^2)+10)(x^2-1)^2(1+z)^2}{1+t}-20$$



Solution of the following auxiliary problem:

$$U_t-\left(\frac{K_H}{R^2}(1-x^2)U_x\right)_x-K_VU_{zz}+\omega U_z=\mathbf{\Phi(x,z,t)}\quad \text{in } (0,T)\times\Omega,$$

$$\omega xU_x+K_VU_z=0\quad \text{in } (0,T)\times\Gamma_H,$$

$$Du_t-\frac{DK_{H_0}}{R^2}\left((1-x^2)^{3/2}|u_x|u_x\right)_x+K_V\frac{\partial U}{\partial n}+\omega xU_x+C+Bu=$$

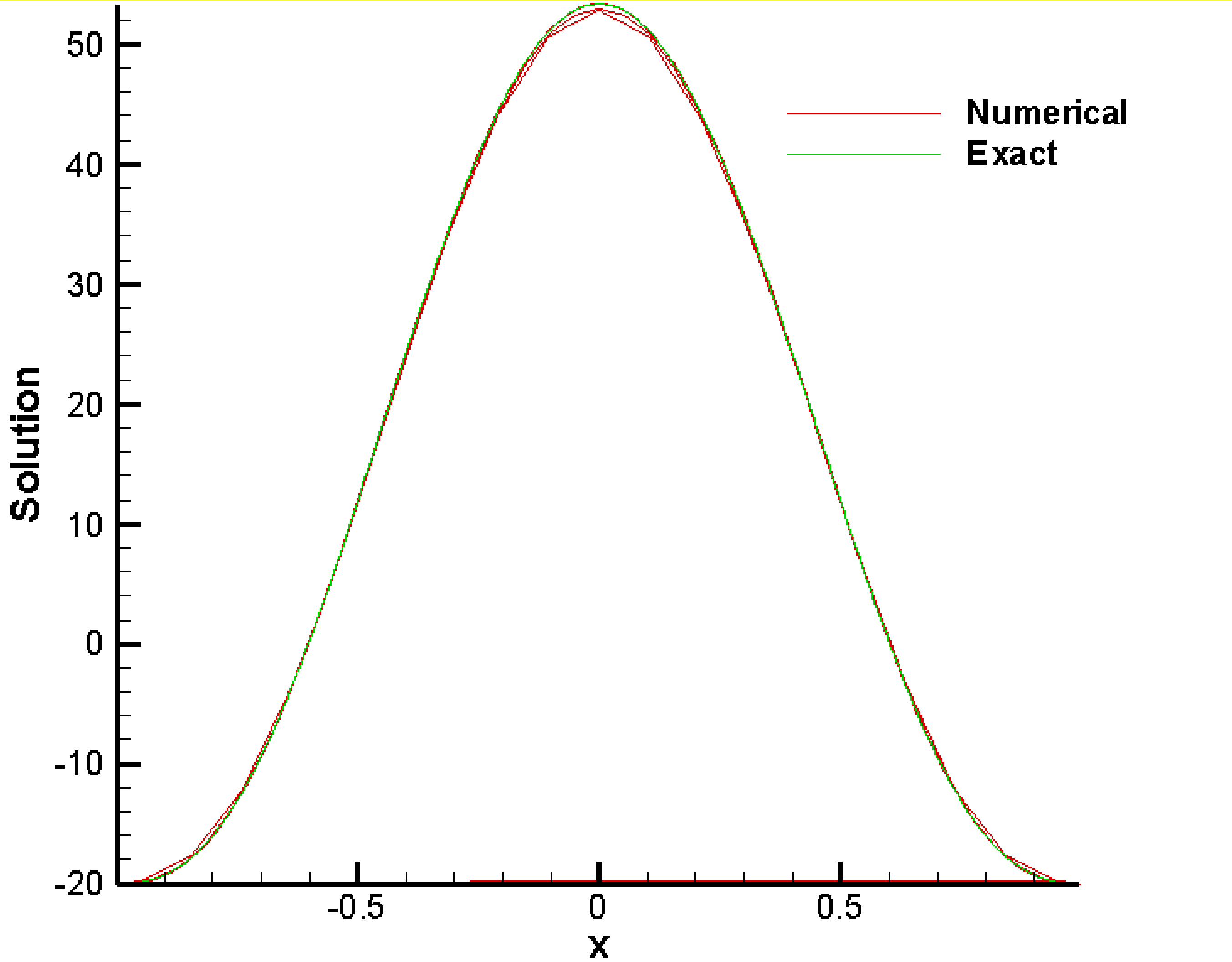
$$=\frac{1}{\rho c}QS(x)\beta(x,U)+\boldsymbol{\psi(x,t)}\quad \text{in } (0,T)\times\Gamma_0,$$

$$(1-x^2)^{3/2}|U_x|U_x=0\quad \text{in } (0,T)\times\Gamma_{-1}\cup(0,T)\times\Gamma_1,$$

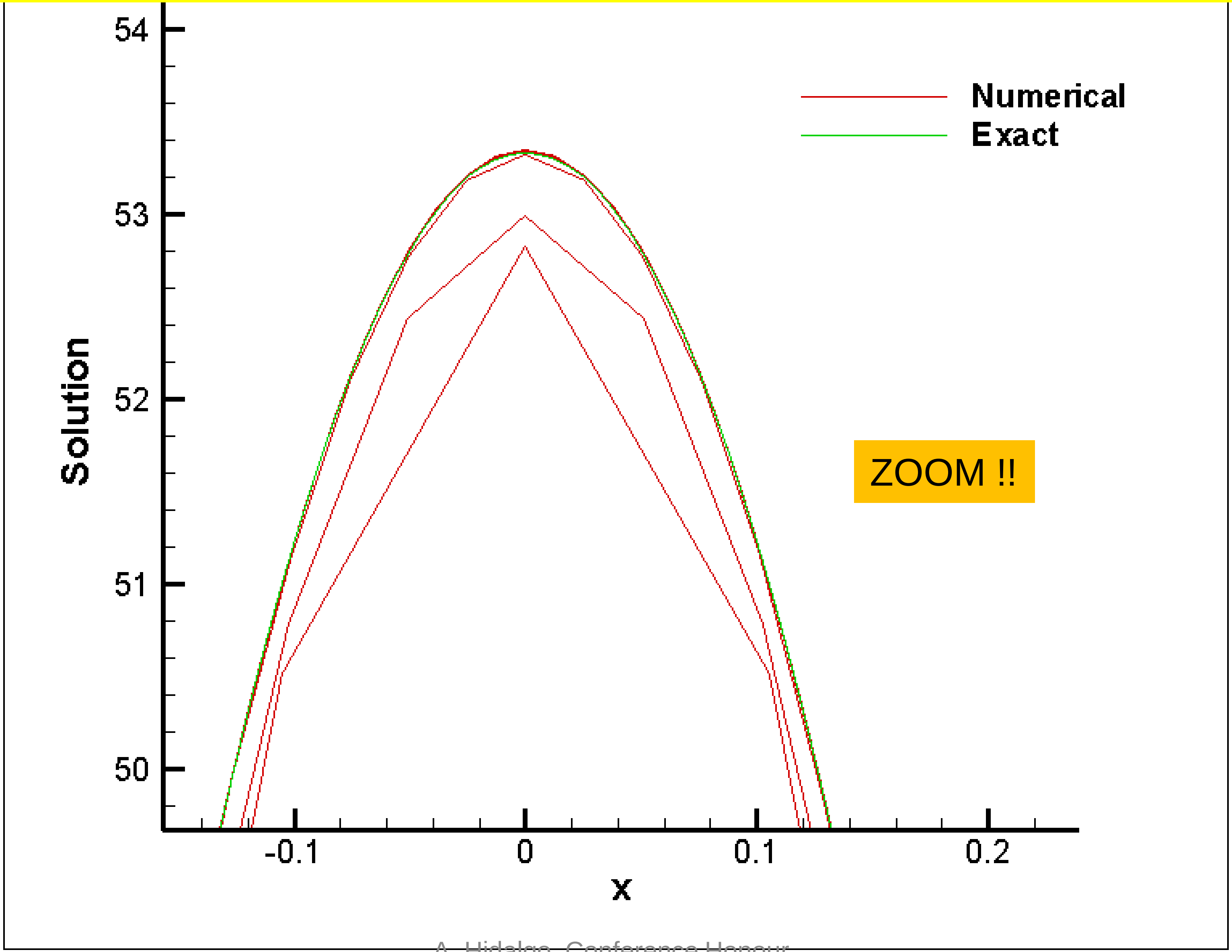
$$U(x,z,0)=(100(1-x^2)+10)(x^2-1)^2(1+z)^2-20\quad \text{in } \Omega,$$

$$u(x,0)=(100(1-x^2)+10)(x^2-1)^2-20\quad \text{in } \Gamma_0,$$

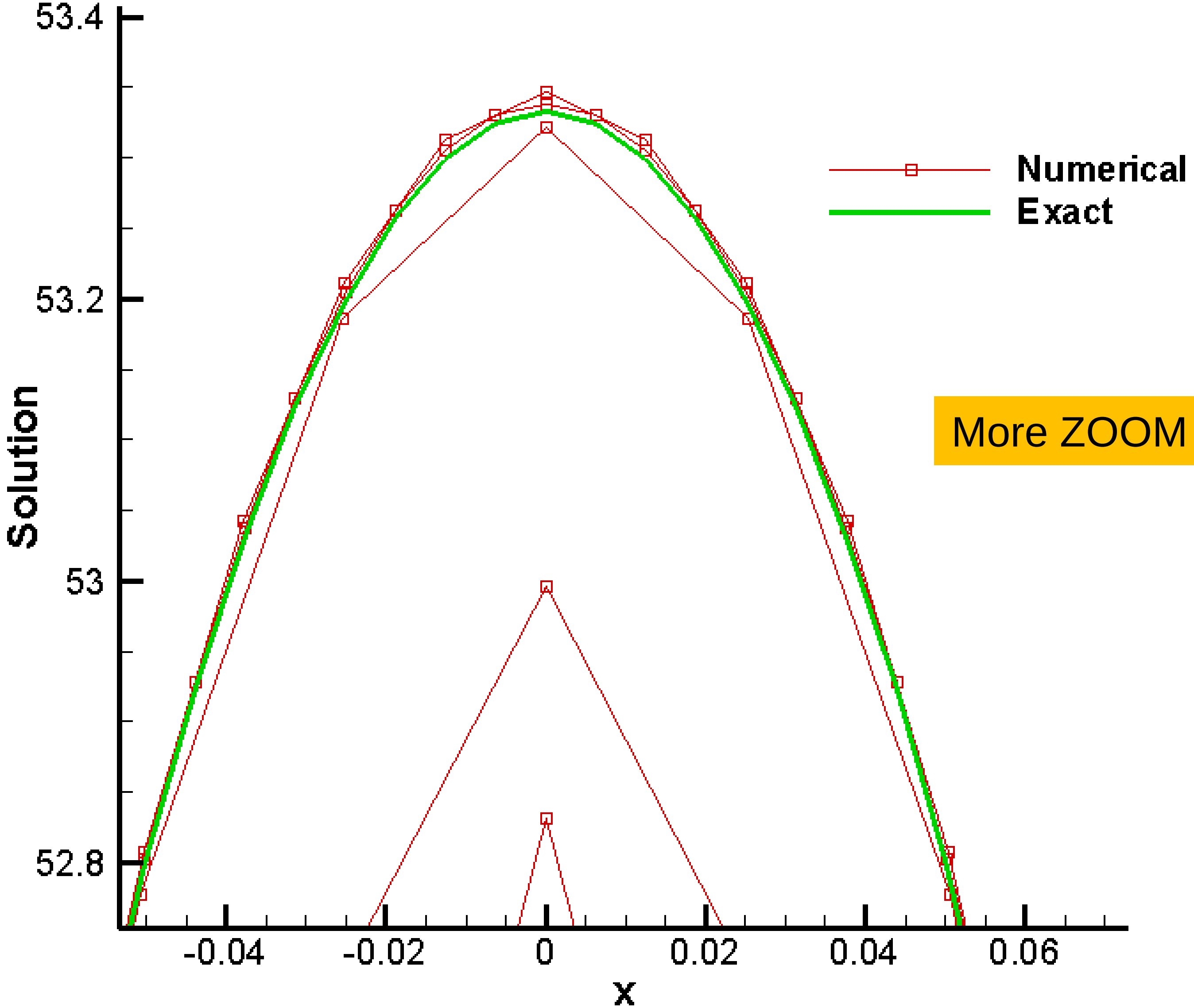
Upper boundary: Exact solution vs. Numerical obtained with 5 meshes (20x10, 40x20, 80x40, 160x80, 320x160



Upper boundary: Exact solution vs. Numerical obtained with 5 meshes (20x10, 40x20, 80x40, 160x80, 320x160)

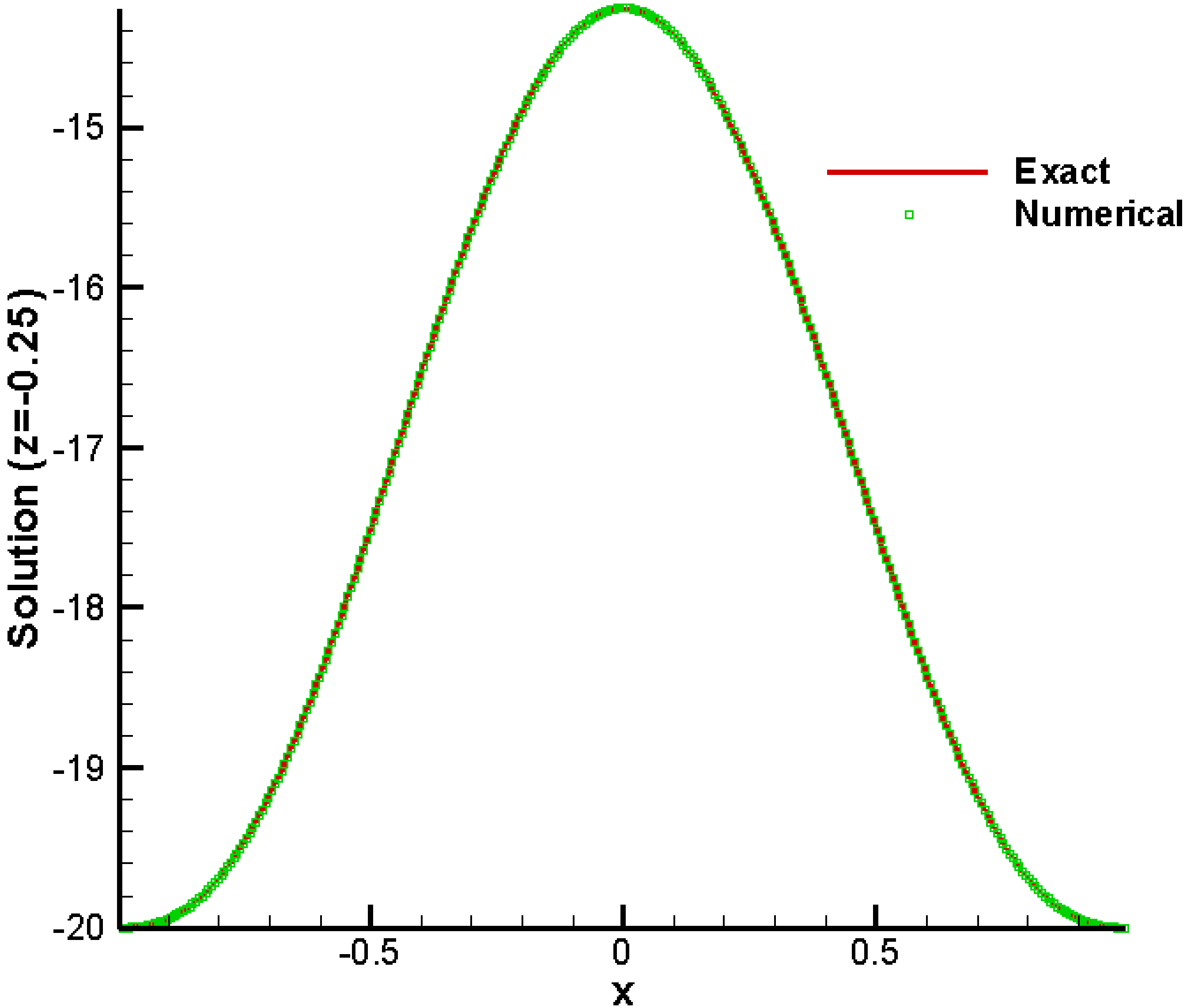


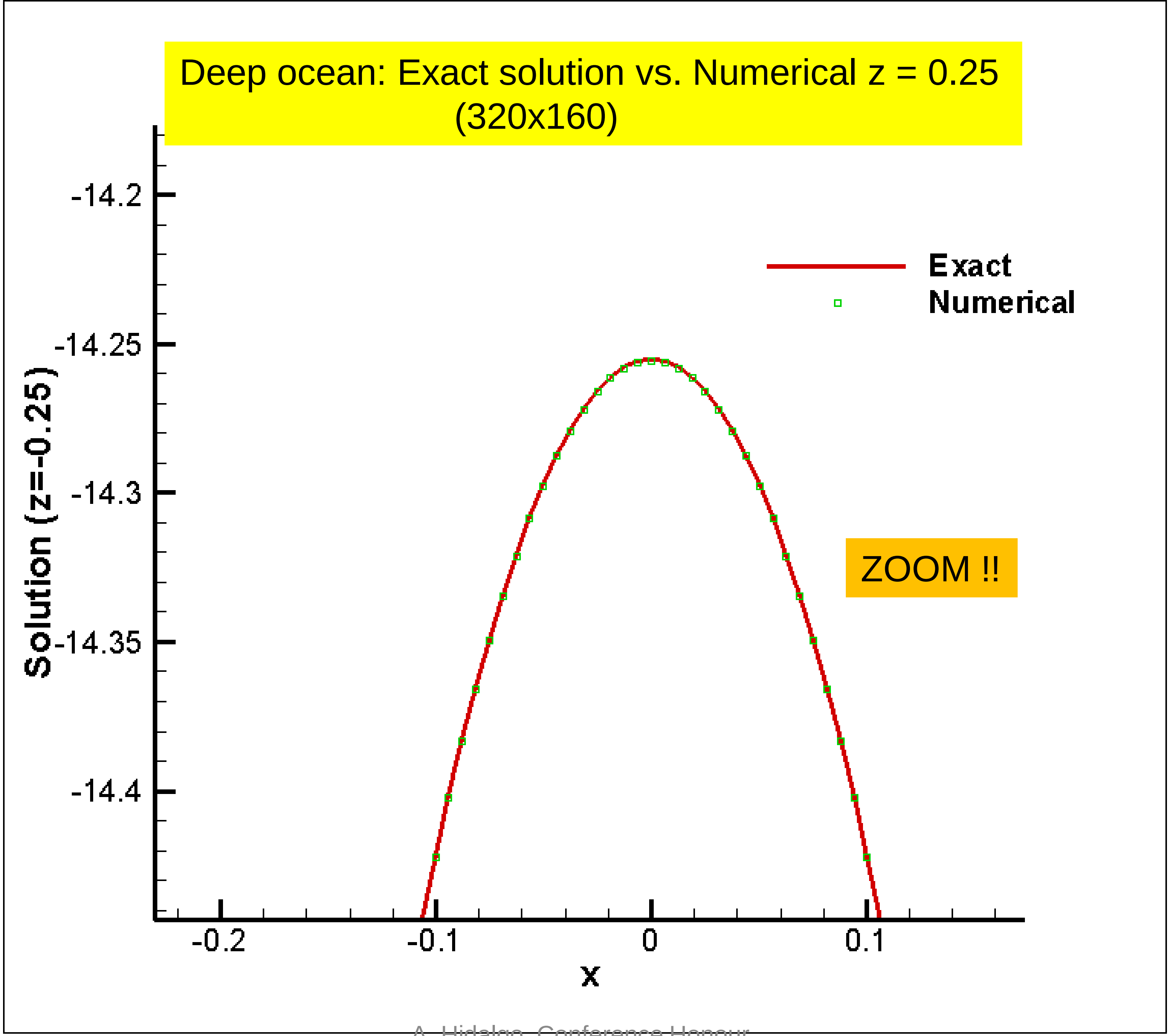
Upper boundary: Exact solution vs. Numerical obtained with 5 meshes (20x10, 40x20, 80x40, 160x80, 320x160)



More ZOOM !!

Deep ocean: Exact solution vs. Numerical z = 0.25 (320x160)

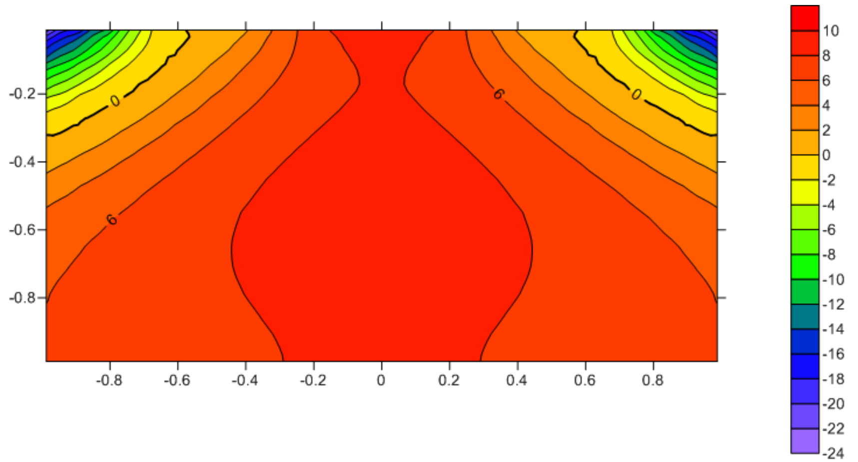


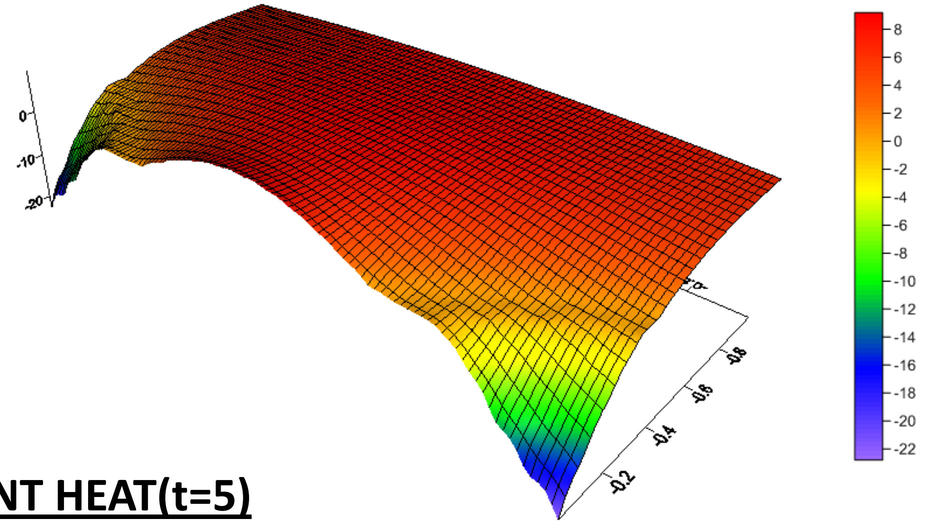

Deep ocean: Exact solution vs. Numerical z = 0.25
(320x160)
Solution (z=-0.25)
-14.2
-14.25
-14.3
-14.35
-14.4
-0.2
-0.1
0
0.1
x
Exact
Numerical
ZOOM !!

Mesh	Cells	$\|\| \text{Error} \|\|_2$
1	20×10	0.4017
2	40×20	0.1824
3	80×40	2.089×10^{-2}
4	160×80	3.098×10^{-3}
5	320×160	1.139×10^{-3}

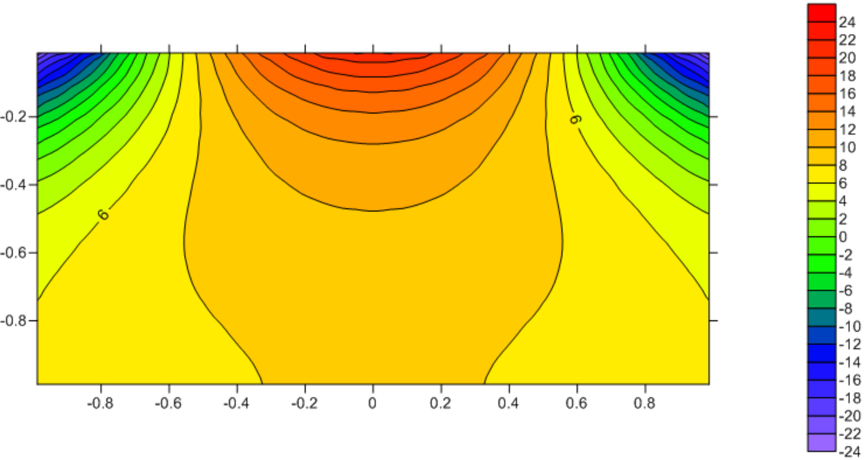
COMPARISON OF RESULTS WITH AND WITHOUT LATENT HEAT

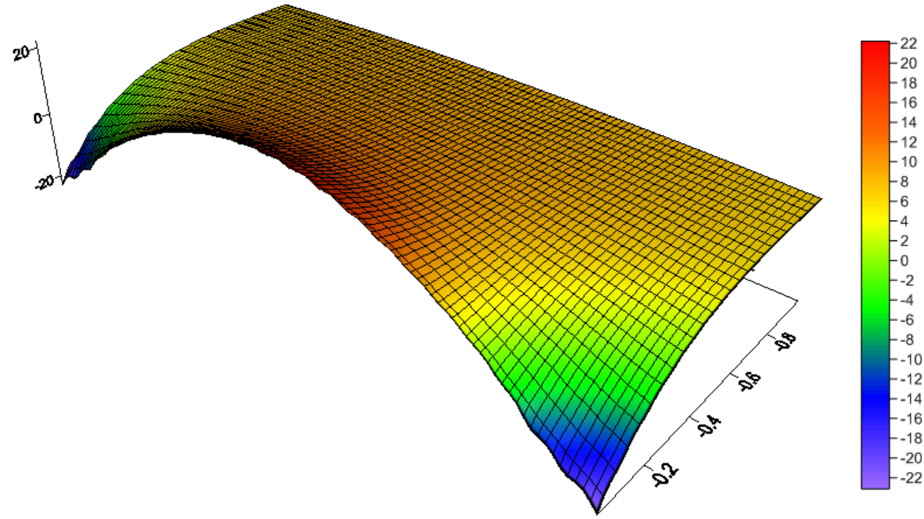
WITH LATENT HEAT (t=5)



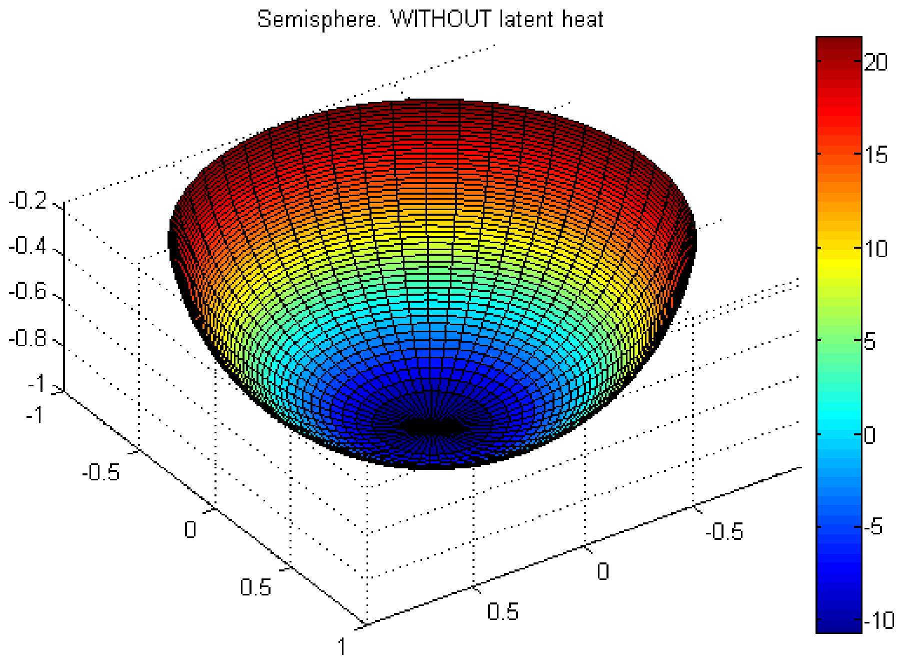


WITHOUT LATENT HEAT(t=5)

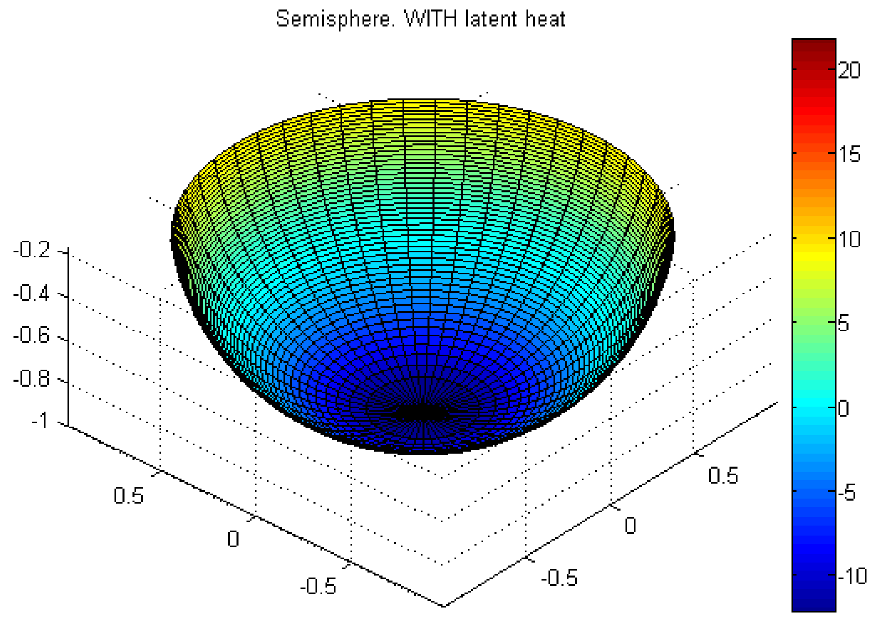


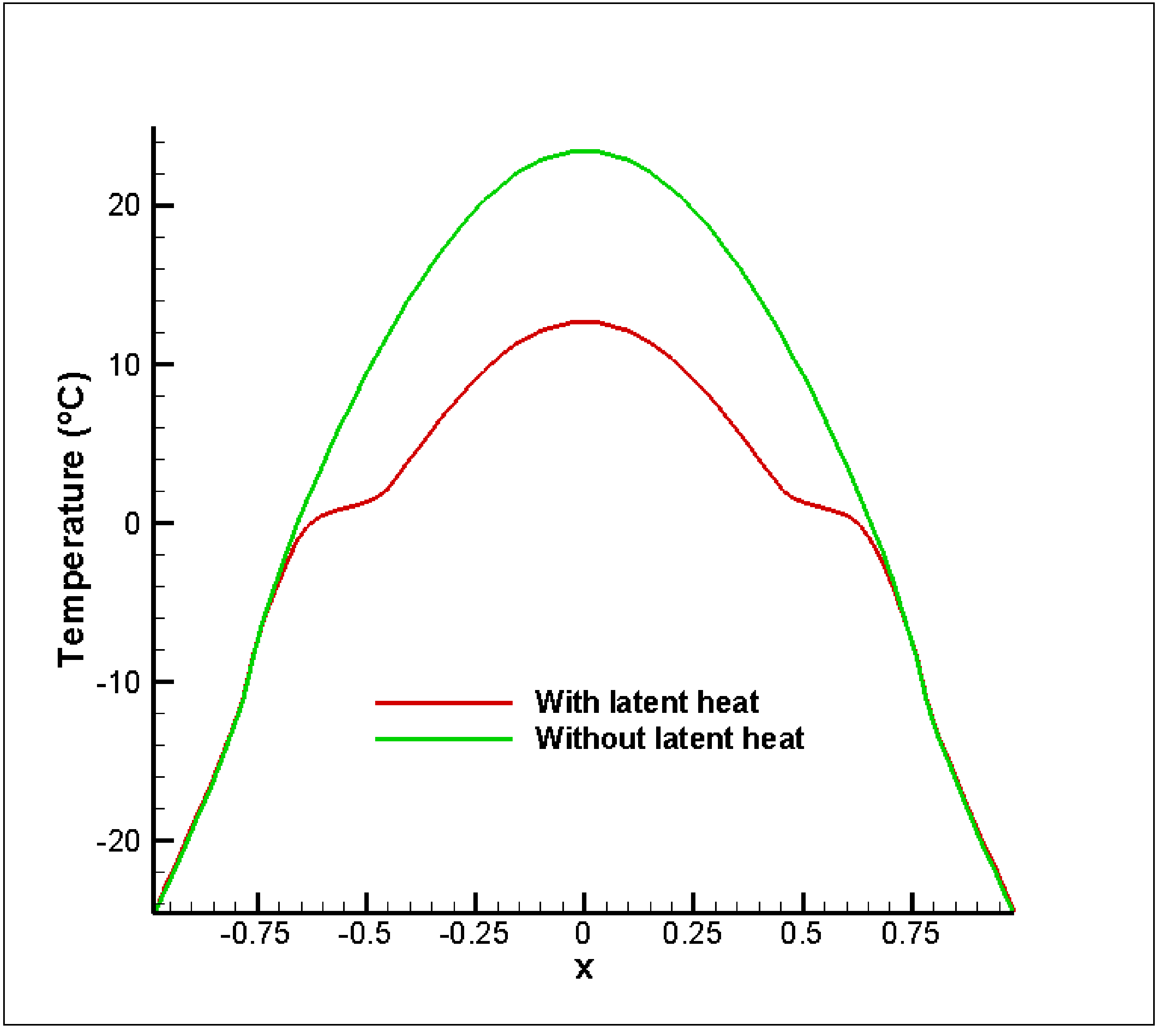


WITHOUT



WITH




Temperature (°C)
20
10
0
-10
-20
-0.75
-0.5
-0.25
0
0.25
0.5
0.75
x
With latent heat
Without latent heat

Conclusions and further research

•We have obtained the numerical solution of a 1D energy balance model with nonlinear diffusion, coupled with a 2D deep ocean model in a rectangular domain.

•The method used is a finite volume method with 3rd order Runge-Kutta TVD.

•It has been obtained the evolution of the temperature in the deep ocean and also in the surface, due to the combination of melting ice, heating-cooling of the surface of the ocean.

•The results show the thermostatic effect of the ocean.

•A verification of the accuracy of the scheme has been carried out solving an auxiliary problem with known analytical solution.

•The effect of the latent heat has been considered in the problem.

•Incorporation of continents.

•More “realistic” values of parameters.

•3D extension.